Exponential Time Hypothesis
part 1

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Introduction

- We already know that W[1]-hard problems are different than the ones which are FPT.
- However, within the FPT class we have different dependency on the parameter in best known algorithms: $2^{\mathcal{O}(\sqrt{k})}, 2^{\mathcal{O}(k)}, 2^{\mathcal{O}(k \log k)}, 2^{2^{\mathcal{O}(k)}}$.
- Similarly for W[1]-hard problems: $f(k)n^{\mathcal{O}(k)}, f(k)n^{\mathcal{O}(\sqrt{k})}, f(k)n^{\mathcal{O}(\log \log k)}$.

Introduction

- It seems that there are some inner hierarchies of subclasses, corresponding to different classes of running times.
- To separate these classes, we need a methodology for proving lower bounds.
- Ideally, if $2^{f(k)}n^{\mathcal{O}(1)}$ is known, we want $2^{o(f(k))}n^{\mathcal{O}(1)}$ lower bound.
- Dream, for $2^{f(k)}n^{\mathcal{O}(1)}$ algorithm, $2^{(1-\epsilon)f(k)}n^{\mathcal{O}(1)}$ lower bound.
- $P \neq NP$ or even $FPT \neq W[1]$ is (probably) not strong enough to serve as a source of hardness.

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- $P \neq NP$ or even $FPT \neq W[1]$ is (probably) not strong enough to serve as a source of hardness.

If we are pessimistic enough, sometimes dreams come true.

What this lecture is about?

Exponential Time Hypothesis (ETH)

Informally: 3-CNF SAT cannot be solved in time subexponential in the number of variables.

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Outline:

1. Formal definition of ETH.
2. Sparsification lemma.
3. ETH and classic NP-hardness reductions: $2^{o(n)}$, $2^{o(k)}n^{O(1)}$.
4. ETH and planar problems: $2^{o(\sqrt{n})}$, $2^{o(\sqrt{k})}n^{\mathcal{O}(1)}$.
5. ETH and W[1]-hard problems: $f(k)n^{o(k)}$.
6. Glimpse on less common lower bounds.

CNF SAT

A formula on variables $x_1, \ldots, x_n$ in *conjunctive normal form* (CNF):
$\varphi = C_1 \wedge C_2 \wedge \ldots C_m$,
where $C_i = \ell_1^i \vee \ell_2^i \vee \ldots \vee \ell_r^i$
and each ℓ_j^i is a literal, i.e., variable or its negation.

- Example: $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_4) \wedge (\neg x_2)$
- If each clause has at most q literals, we have q-CNF formula.
- CNF-SAT: given a CNF formula decide if it is satisfiable.
- q-SAT: same for q-CNF.

CNF SAT

- 3-SAT is NP-complete, hence no poly time algorithm,
- exhaustive search gives $\mathcal{O}^*(2^n)$ time algorithm for CNF-SAT,

  $\mathcal{O}^*()$ notation suppresses polynomial factors
- q-SAT is solvable in $\mathcal{O}^*(2^{\gamma_q n})$ where $\gamma_q = 1 - \Theta(1/q)$,
- current best algorithm for 3-SAT: $\mathcal{O}(2^{0.386n})$.

Current status exposes two barriers hard to break:

1. $2^{o(n)}$ algorithm for 3-SAT,
2. $\mathcal{O}^*((2-\epsilon)^n)$ algorithm for general CNF-SAT for some $\epsilon > 0$.

ETH and SETH

Definition

For $q \geqslant 3$ let δ_q be infimum of the set
$\{c : \text{there is } \mathcal{O}^*(2^{cn}) \text{ algorithm for q-SAT}\}$

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Conjectures

- Exponential Time Hypothesis (ETH): $\delta_3 > 0$.
- Strong Exponential Time Hypothesis (SETH): $\lim_{q\to\infty} \delta_q = 1$.

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Why not the following?

- ETH: no $2^{o(n)}$ algorithm for 3-SAT,
- SETH: no $\mathcal{O}^*((2-\epsilon)^n)$ algorithm for CNF-SAT for any $\epsilon > 0$.

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- One could define $\delta_q^* \leqslant \delta_q$ as infimum over randomized two-sided error algorithms.

Two-sided error algorithms

A randomized algorithm with two-sided error is an algorithm which for any YES-instance and any NO-instance returns an incorrect answer with prob. at most constant $p < 1/2$.

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- Reasonable from the point of view of research on satisfiability.
- But inconvenient from pragmatic point of view: dealing with error probability in reductions.
- Usually for a chain of reductions excluding some deterministic algorithm, the same chain works for randomized algorithms.
- During this school we assume deterministic versions (discussion on randomized versions in the book).

Plausibility

- ETH is generally considered as plausible.
- Many believe that SETH can be refuted anytime.
- Still, SETH based lower bounds make sense: existence of better algorithm would constitute a major breakthrough in the complexity of satisfiability.
- In other words, if you do not believe in a lower bound X which is SETH based, then refute SETH first before trying to disprove X.

Take-home message

SETH $\implies$ ETH $\implies$ W[1] $\neq$ FPT $\implies$ P $\neq$ NP

FPT $\neq$ W[1] $\implies$ no $\mathcal{O}^*(f(k))$
ETH $\implies$ no $\mathcal{O}^*(2^{o(f(k))})$
SETH $\implies$ no $\mathcal{O}^*(2^{(1-\epsilon)f(k)})$.

Sparsification Lemma

Consider a reduction

Given a 3-SAT formula φ with n vars and m clauses, produces equivalent instance of problem X of size $\mathcal{O}(n+m)$.

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Proof by contradiction:

- Assume there is such an algorithm $\mathcal{A}$.
- For a given 3-SAT formula φ: (i) apply the reduction, (ii) use $\mathcal{A}$.
- Pipelined algorithm works in $2^{o(|I|^{1/3})} = 2^{o((n+m)^{1/3})} = 2^{o(n)}$.

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- Pipelined algorithm works in $2^{o(|I|^{1/3})} = 2^{o((n+m)^{1/3})} = 2^{o(n)}$.

If we knew that $m = \mathcal{O}(n)$, we would get a much stronger lower bound of $\mathcal{O}^*(2^{o(|I|)})$.

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Sparsification Lemma

For any $q \geqslant 3$ and $\epsilon > 0$ there is a constant $C = C(q, \epsilon)$ such that any q-CNF formula φ can be expressed as $\varphi = \bigvee_{i=1}^{t} \psi_i$, where $t \leqslant 2^{\epsilon n}$ and each ψ_i is a q-CNF formula with the same set of variables and at most $C \cdot n$ clauses. Such disjunction can be computed in time $\mathcal{O}^*(2^{\epsilon n})$.

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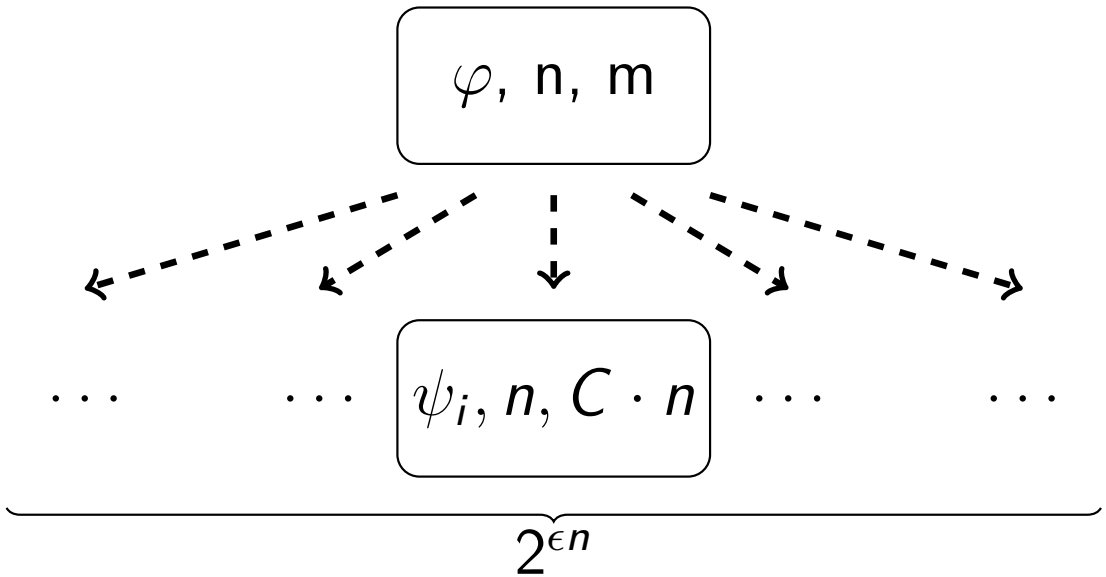

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Unless ETH fails, there is a constant $c > 0$, such that no algorithm solves 3-SAT in time $\mathcal{O}^*(2^{c(n+m)})$.

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Proof by contradiction:

- Assume for every $c > 0$ there is an algorithm $\mathcal{A}_c$ solving 3-SAT in $\mathcal{O}^*(2^{c(n+m)})$.
- Consider any $d > 0$. We want to solve 3-SAT in time $\mathcal{O}^*(2^{dn})$.
- Use Sparsification Lemma for $\epsilon = d/2$. Denote $C = C(3, \epsilon)$.
- Solve each ϕ_i by $\mathcal{A}_{c'}$, where $c' = \frac{d}{2(C+1)}$.
- The total running time is $\mathcal{O}^*(2^{\epsilon}) + \mathcal{O}^*(2^{\epsilon} \cdot 2^{\frac{d}{2(C+1)} \cdot (C+1)n}) = \mathcal{O}^*(2^{dn})$.

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Theorem

Unless ETH fails, there is a constant $c > 0$, such that no algorithm solves 3-SAT in time $\mathcal{O}^*(2^{c(n+m)})$.

Corollary

Unless ETH fails, there is no $2^{o(n+m)}$ time algorithm for 3-SAT.

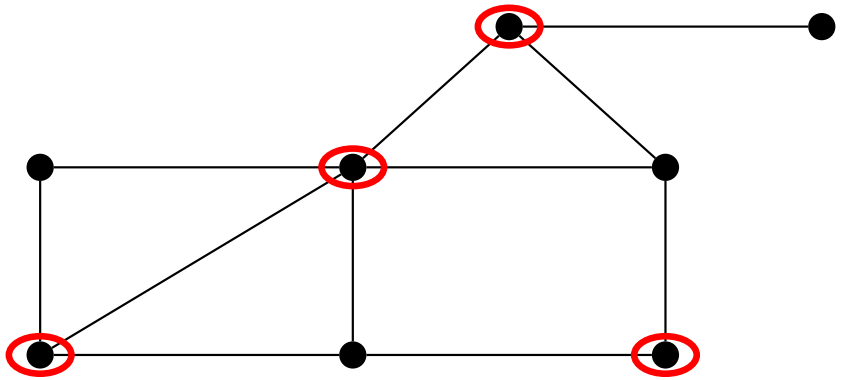
Combining ETH with classic NP-hardness reductions

Vertex Cover

Vertex Cover

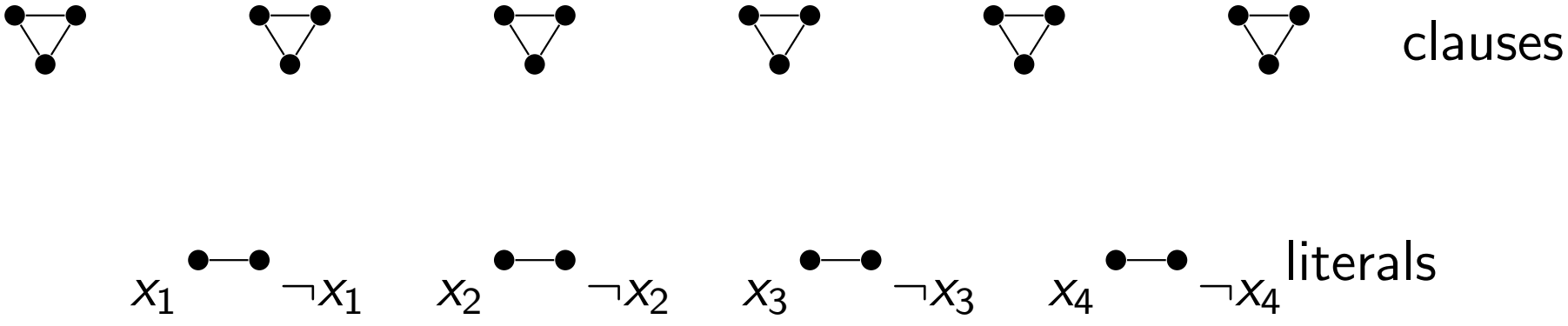
Input: undirected G, integer k

Question: is there a subset $X \subseteq V(G)$ of size at most k, such that for each $uv \in E(G)$ we have $\{u, v\} \cap X \neq \emptyset$.



Vertex Cover

Let us inspect the standard reduction from 3-SAT to Vertex Cover.



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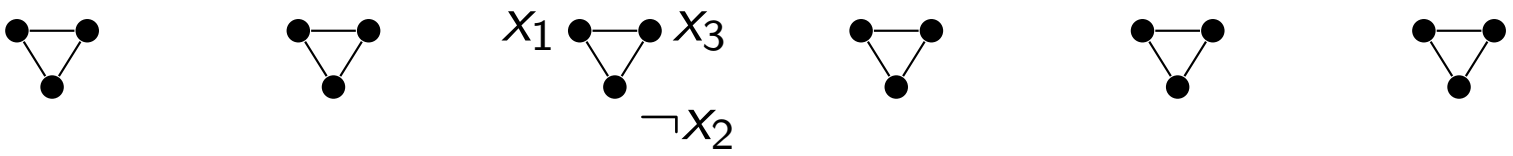

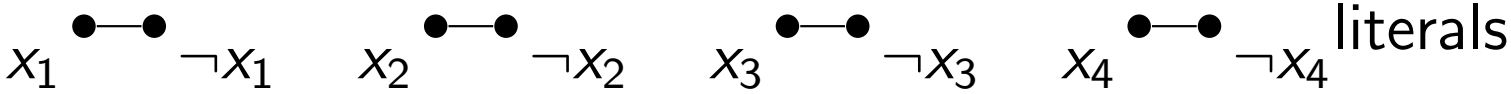

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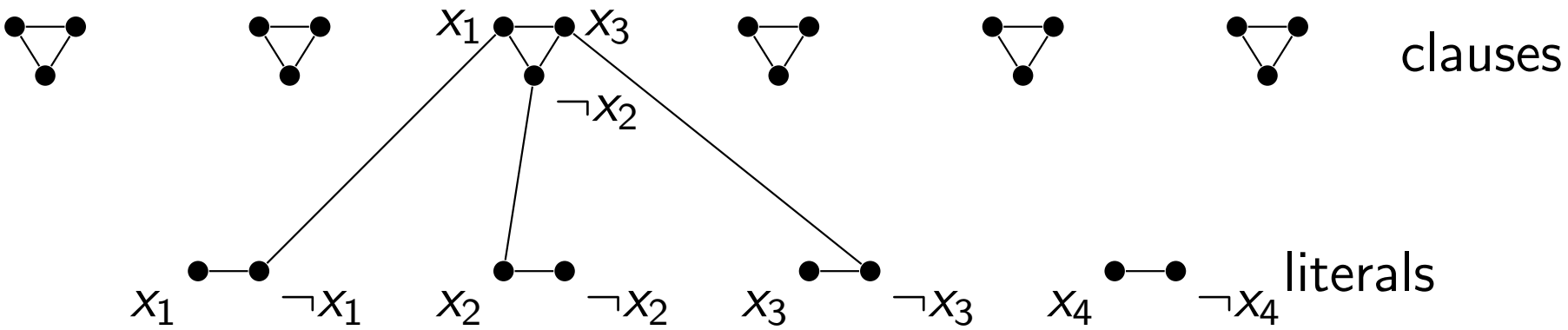

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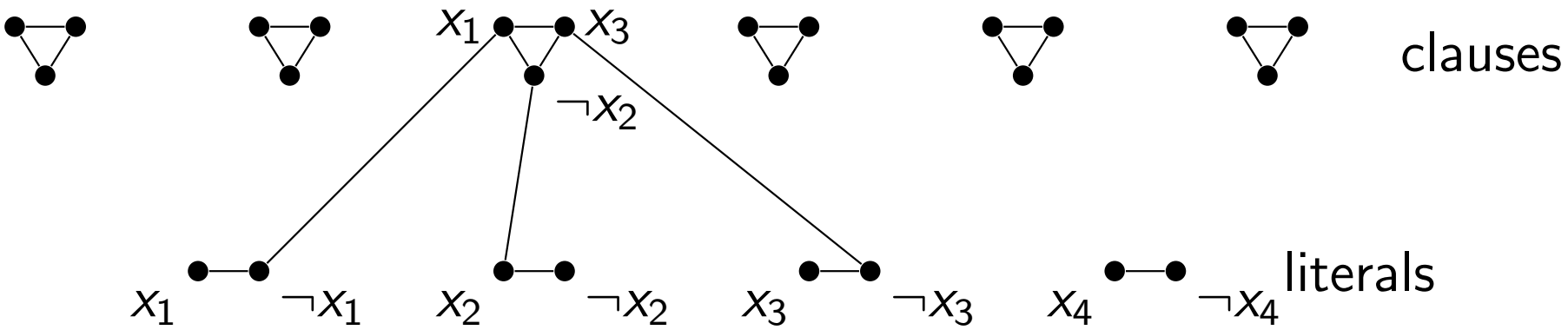

The formula is satisfiable iff the created graph has vertex cover of size $n + 2m$.

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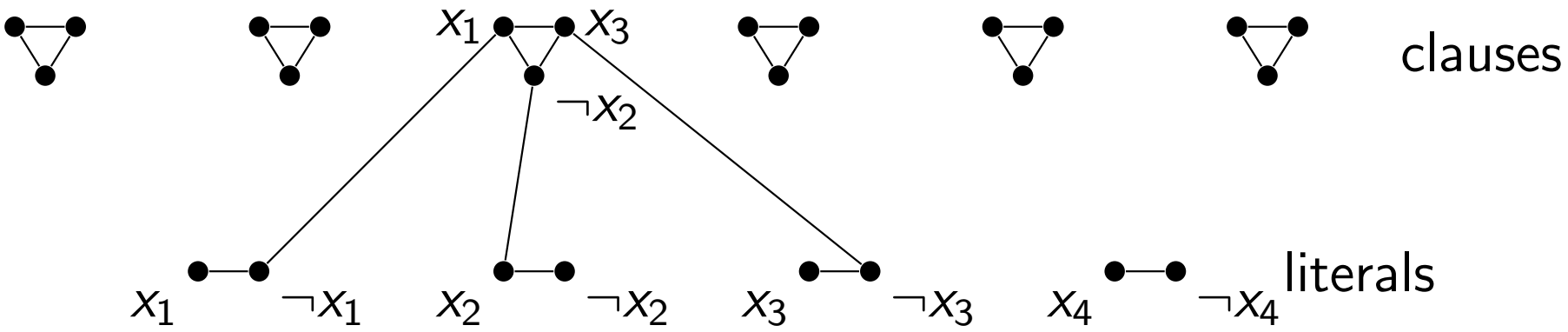

The formula is satisfiable iff the created graph has vertex cover of size $n + 2m$.

Created instance has $O(n + m)$ vertices and edges.

Lemma

Unless ETH fails, there is no $2^{o(n+m)}$ time algorithm for 3-SAT.

Reduction

Given a 3-SAT formula φ with n variables and m clauses one can in polynomial time create a graph with $O(n+m)$ vertices, which has a vertex cover of size $2n+m$ iff φ is satisfiable.

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Corollary

Unless ETH fails, there is no $2^{o(|V(G)|+|E(G)|)}$ time algorithm VC.

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Corollary

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Corollary 2

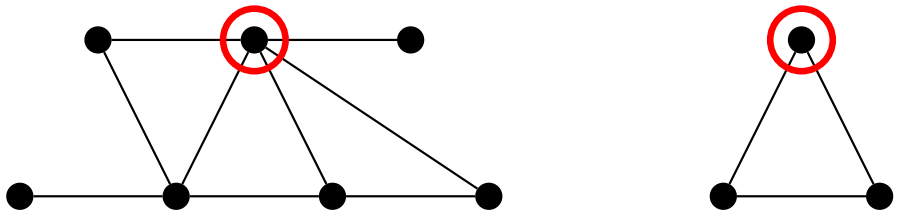
Unless ETH fails, there is no $\mathcal{O}^*(2^{o(k)})$ time FPT algorithm for VC.

FVS

Feedback Vertex Set (FVS)

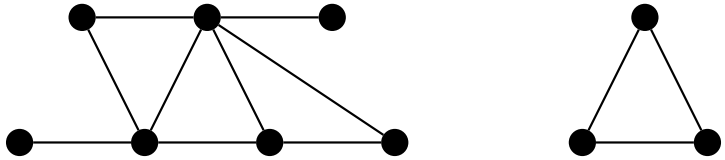
Input: undirected G, integer k

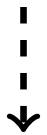
Question: is there a subset $X \subseteq V(G)$ of size at most k, such that $G \setminus X$ is a forest



FVS

VC instance





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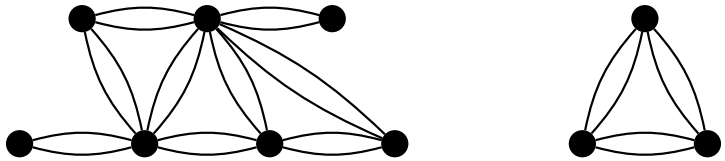

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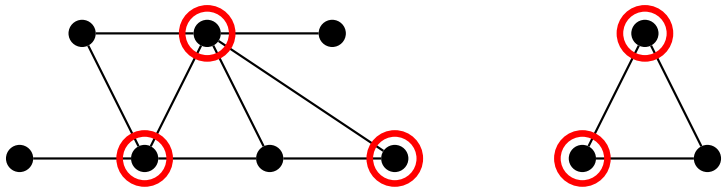

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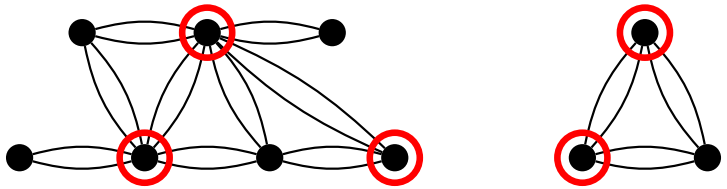

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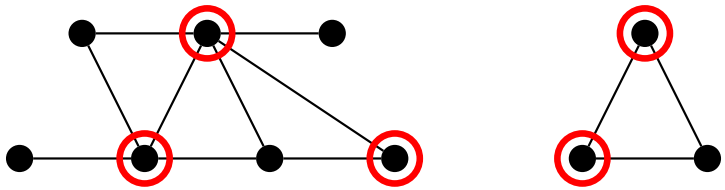

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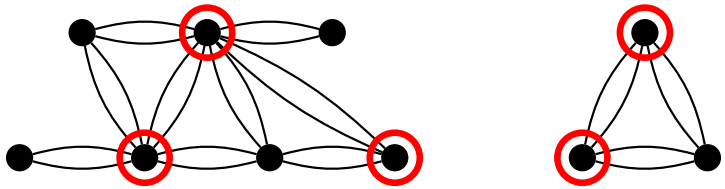

Lemma

Unless ETH fails, there is neither $2^{o(n+m)}$ nor $\mathcal{O}^*(2^{o(k)})$ time algorithm for FVS.

Theorem

Unless ETH fails, there is no $2^{o(n+m)}$ time algorithm for:

- Vertex Cover,
- Feedback Vertex Set,
- Odd Cycle Transversal (exercise),
- Hamiltonian Cycle,
- 3-Coloring (exercise).

Planar lower bounds

Planar Vertex Cover

Input: undirected planar G, integer k

Question: is there a VC of size at most k.

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VC can be solved in time $\mathcal{O}^*(2^{\mathrm{tw}})$,
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Bidimensionality (Wed): planar VC can be solved in time $\mathcal{O}^*(2^{\sqrt{k}})$.

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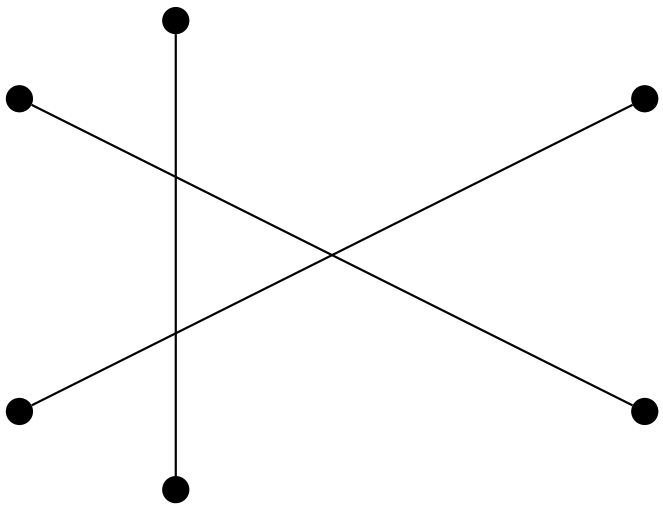

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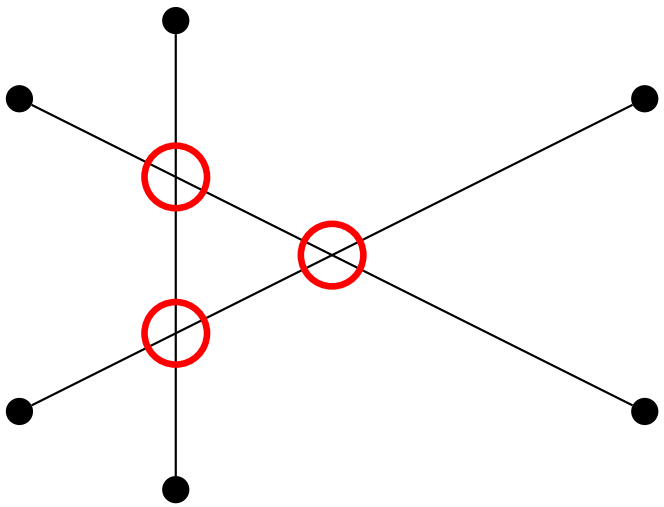

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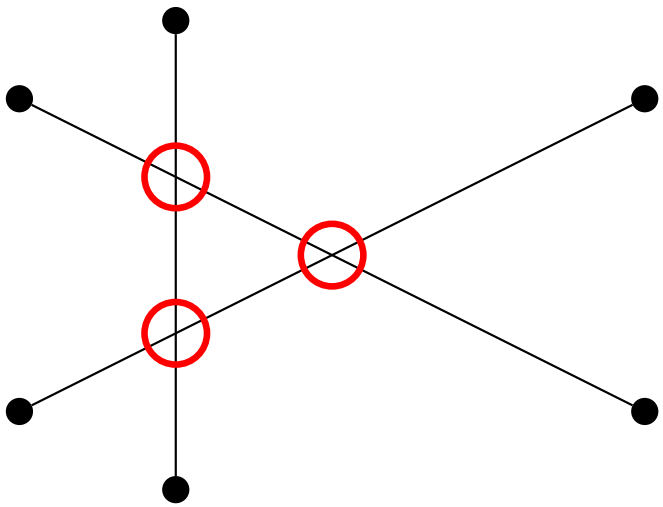

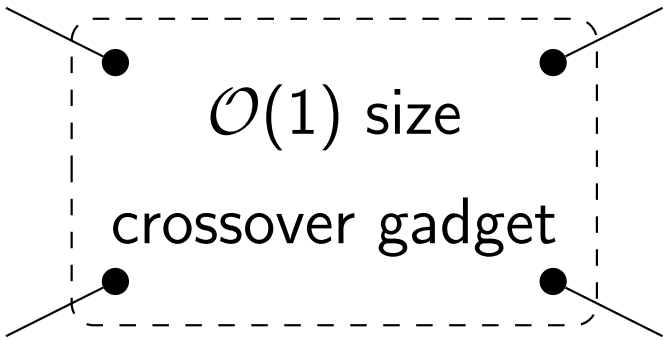

ETH lower bounds methodology

Observation

In order to exclude an algorithm with running time $\mathcal{O}^*(2^{o(f(k))})$, we need to provide a reduction from 3-SAT to B that outputs instances with parameter k bounded by $g(n+m)$, where g is the inverse of f.

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- for $\mathcal{O}^*(2^{o(k)})$ we need $k = \mathcal{O}(n+m)$,
- for $\mathcal{O}^*(2^{o(\sqrt{k})})$ we need $k = \mathcal{O}((n+m)^2)$,
- for $\mathcal{O}^*(2^{o(k^2)})$ we need $k = \mathcal{O}(\sqrt{n+m})$,
- for $\mathcal{O}^*(2^{2^{o(k)}})$ we need $k = \mathcal{O}(\log(n+m))$.

ETH for W[1]-hard problems

Theorem (implies W[1] $\neq$ FPT)

Unless ETH fails, there is no $f(k)n^{o(k)}$ time algorithm for Clique.

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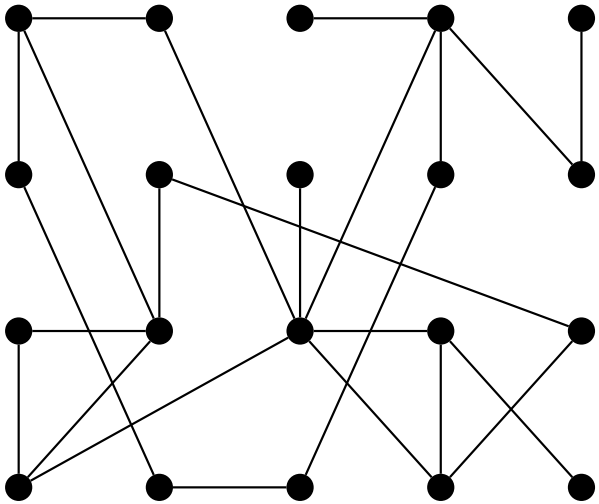
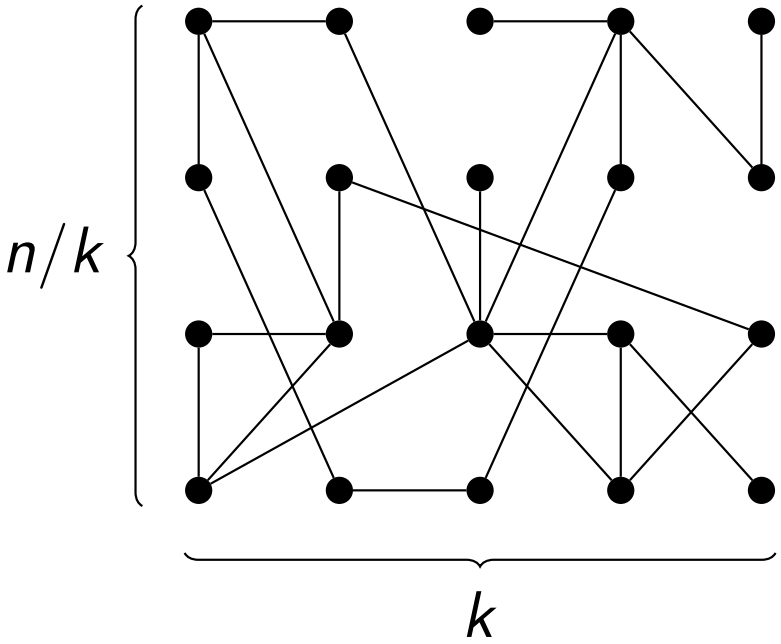
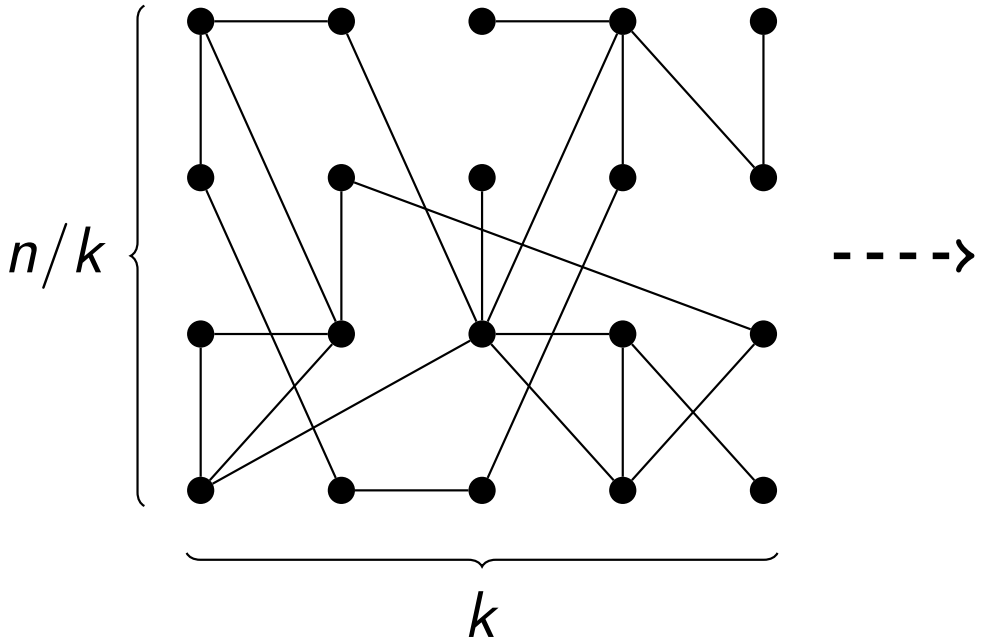
Proof:

- A reduction from 3-Coloring (for which there is no $2^{\mathcal{O}(n+m)}$ time algorithm under ETH).
- In 3-Coloring we have n variables with 3 possible choices each.
- In Clique we have k variables with N possible choices each.
- We want $3^n \tilde{=} N^k$, so $N \tilde{=} 3^{n/k}$.
- But how to choose k? (decide later, but it has to be $\omega(1)$)




n/k
k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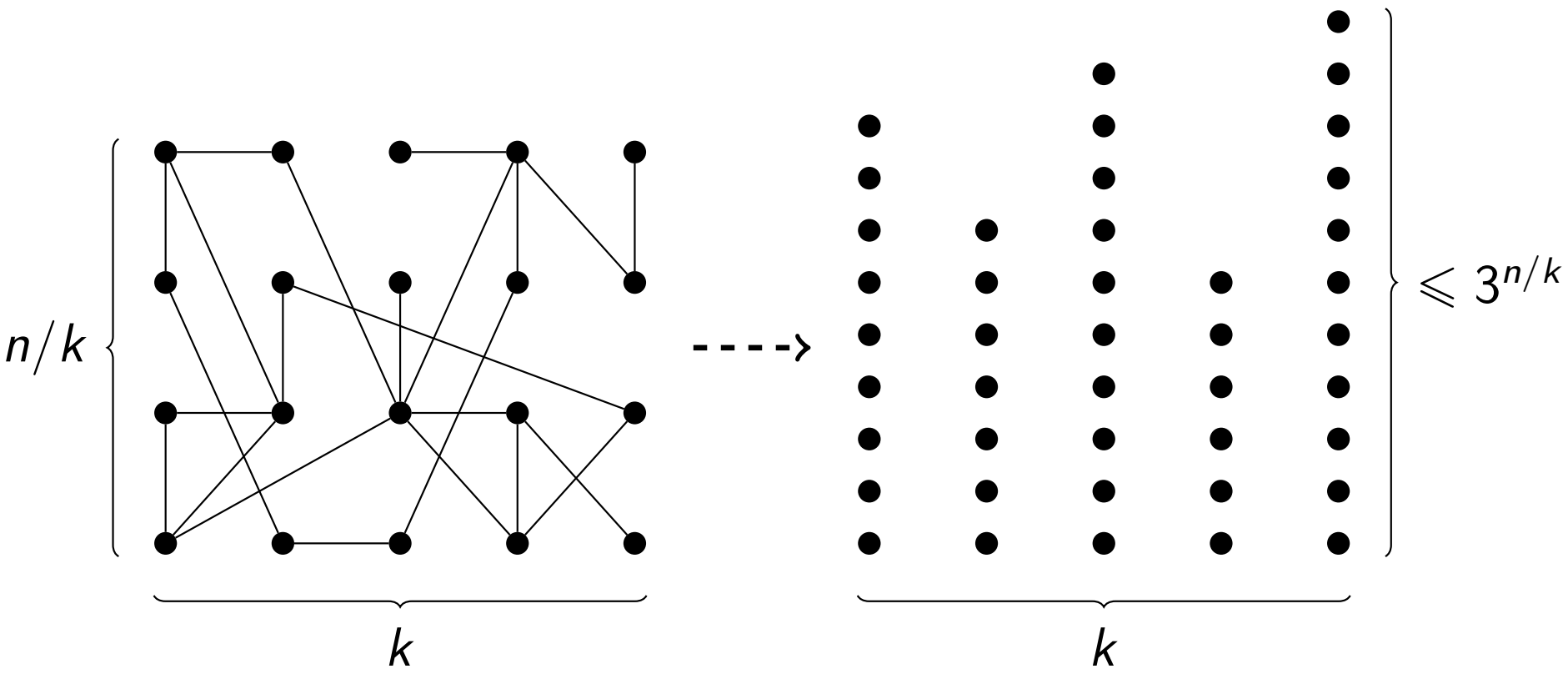
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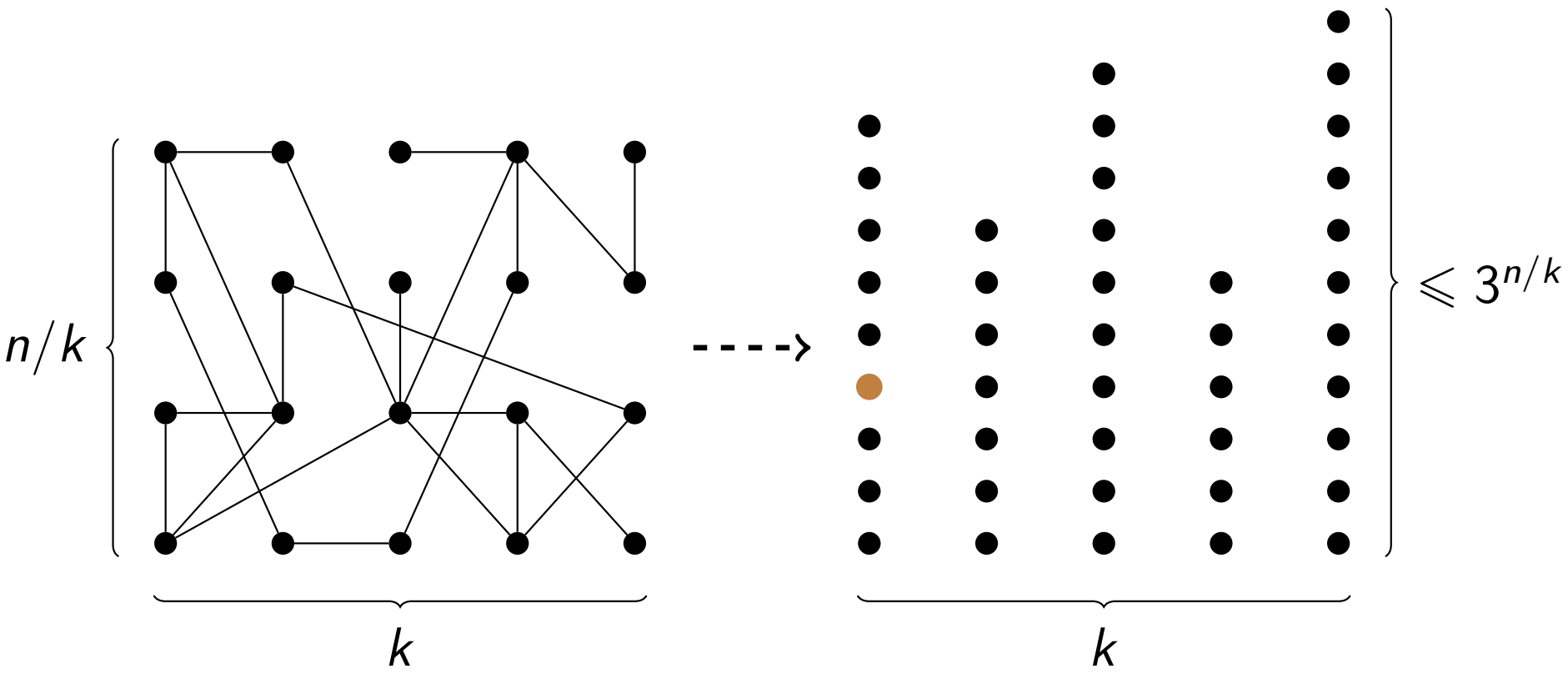
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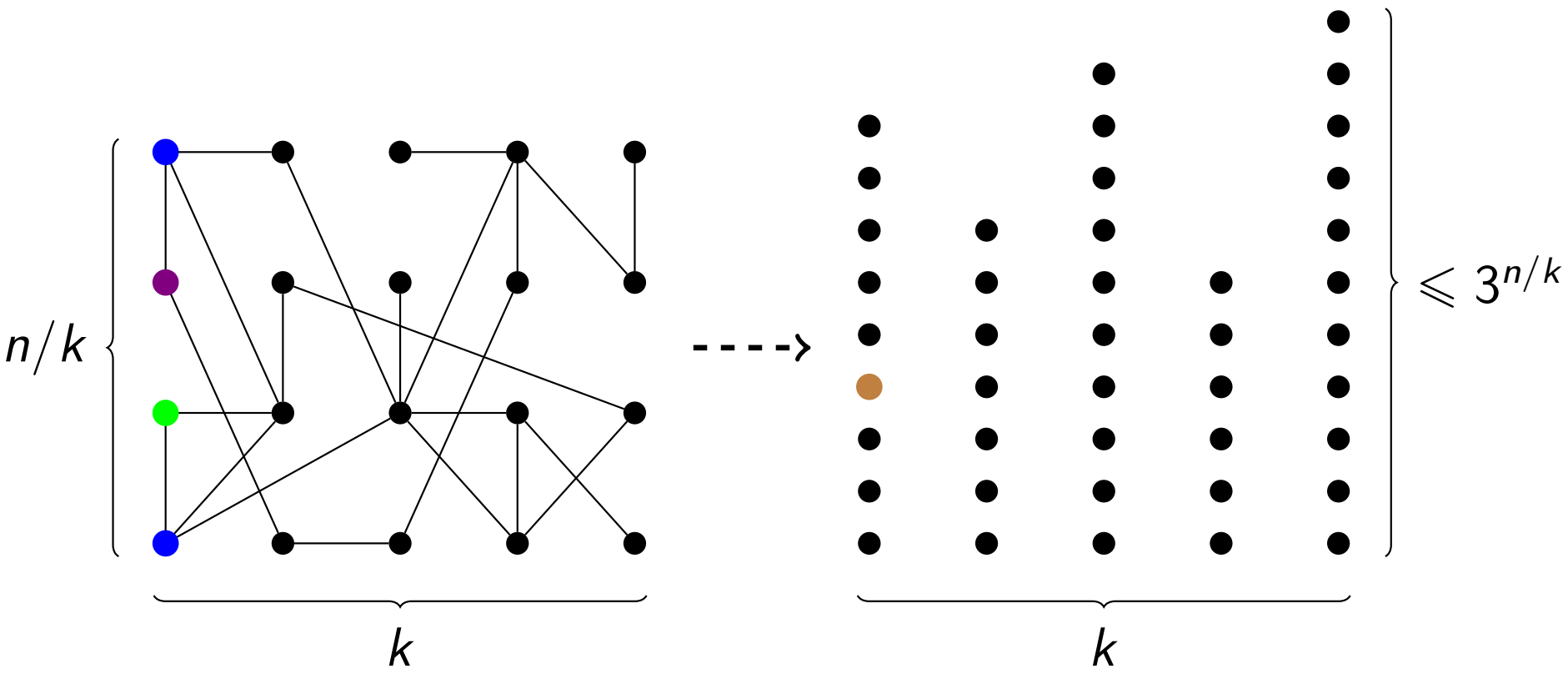
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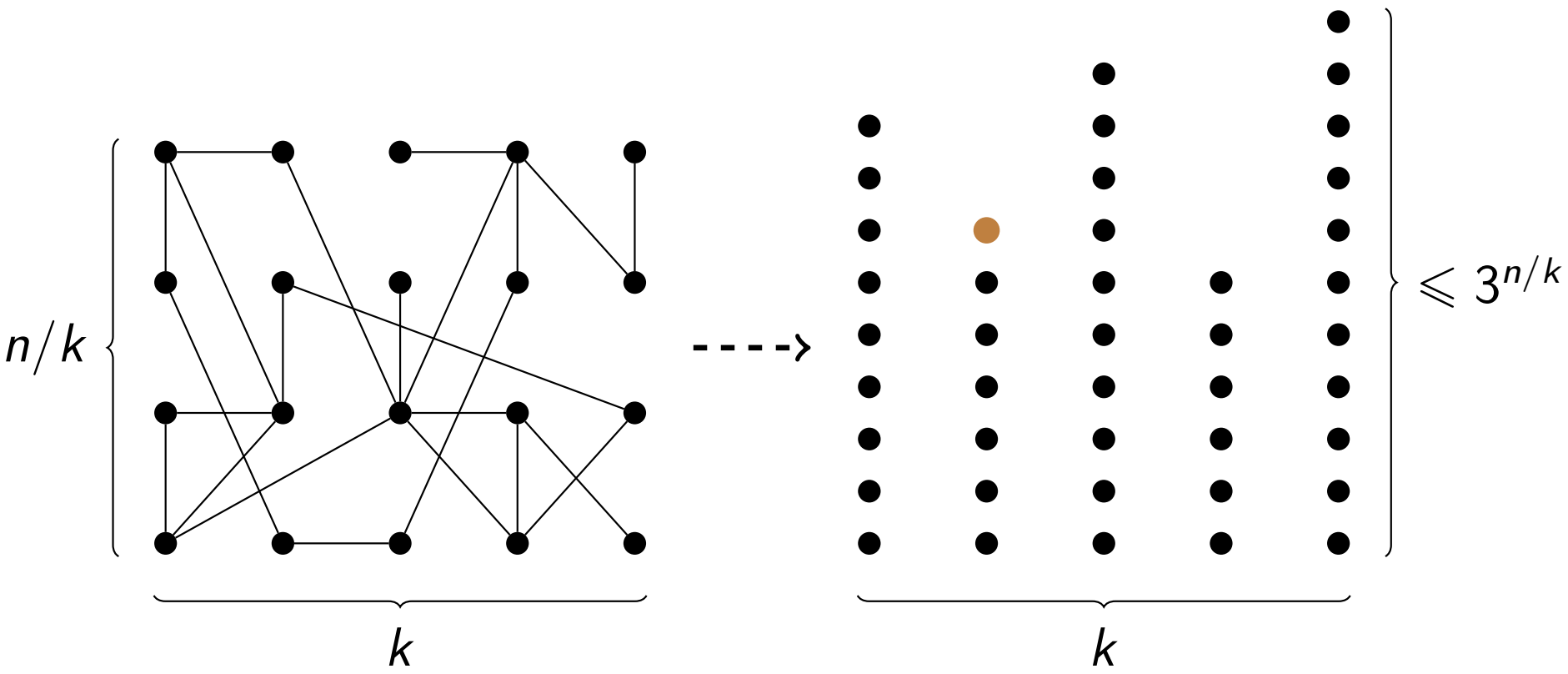
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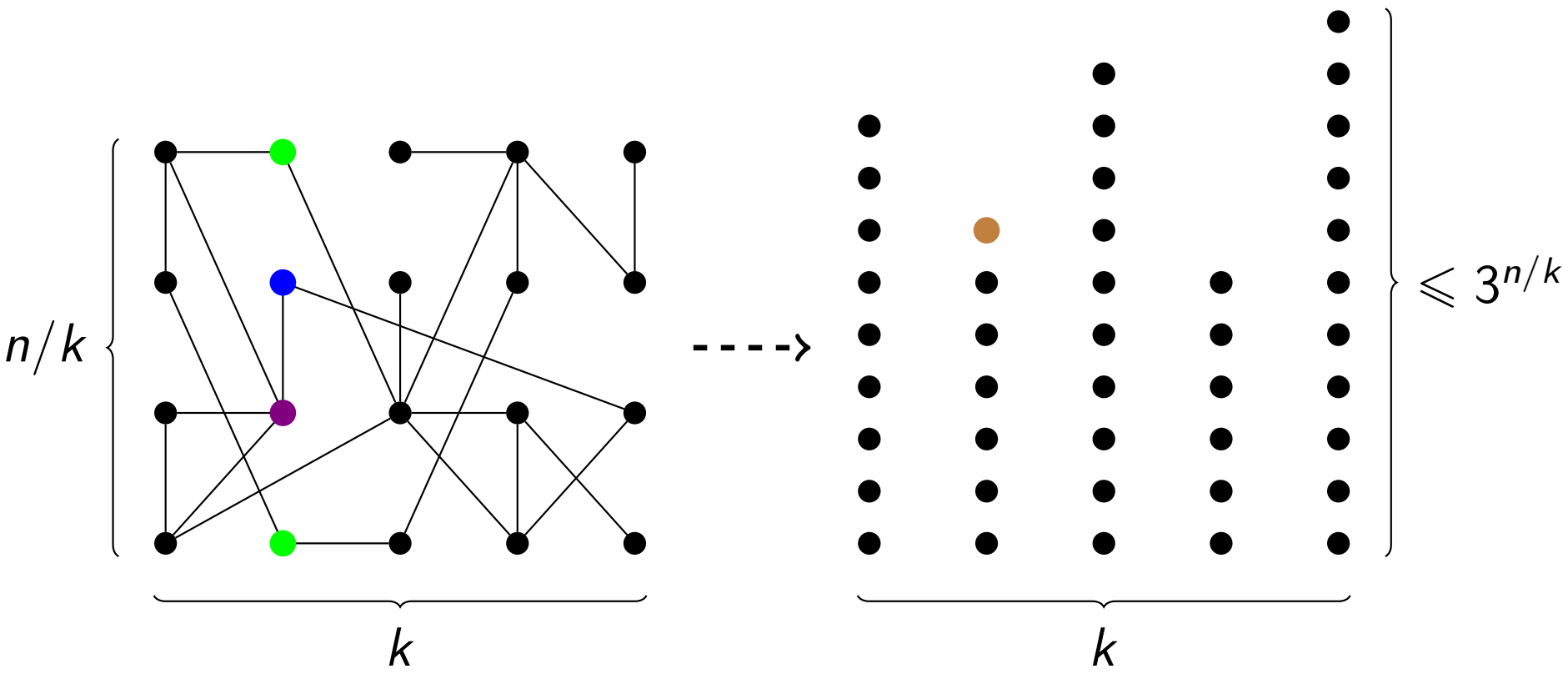
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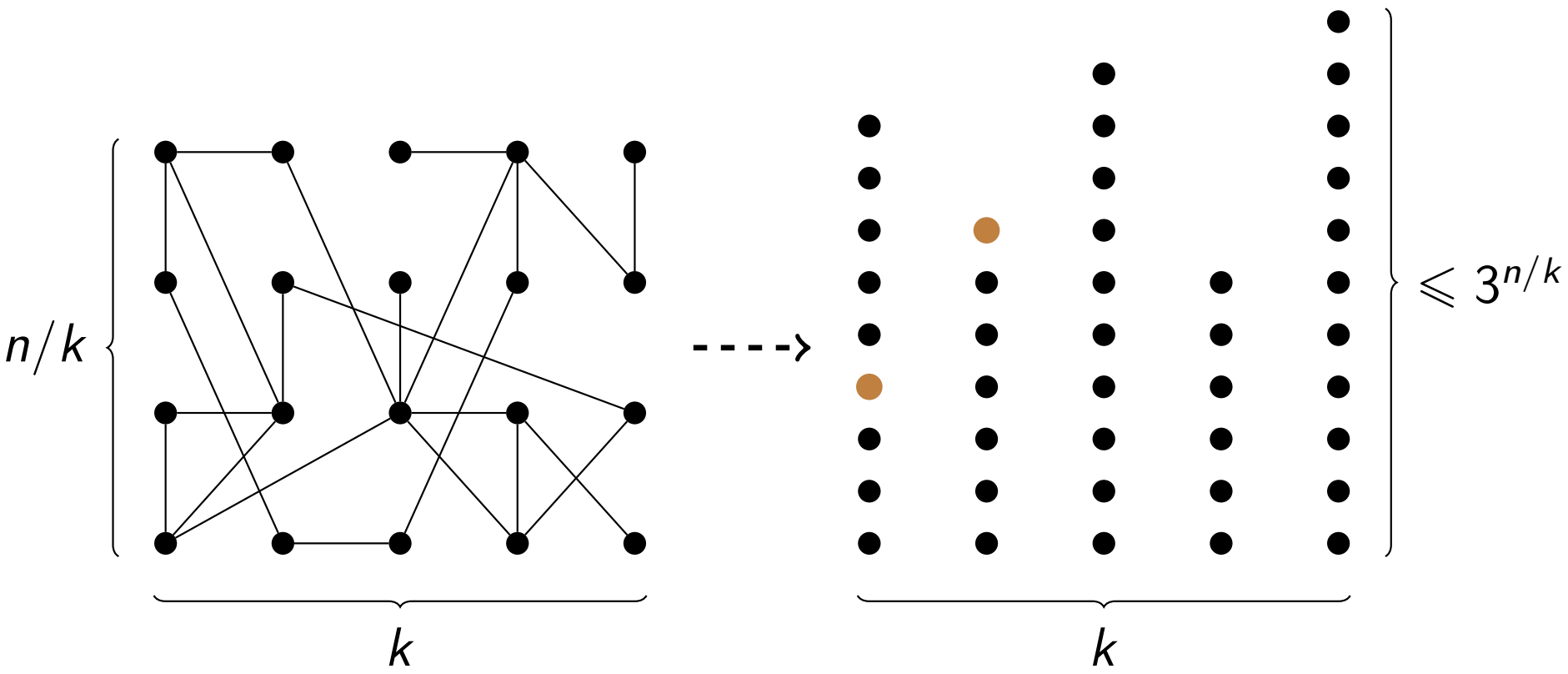
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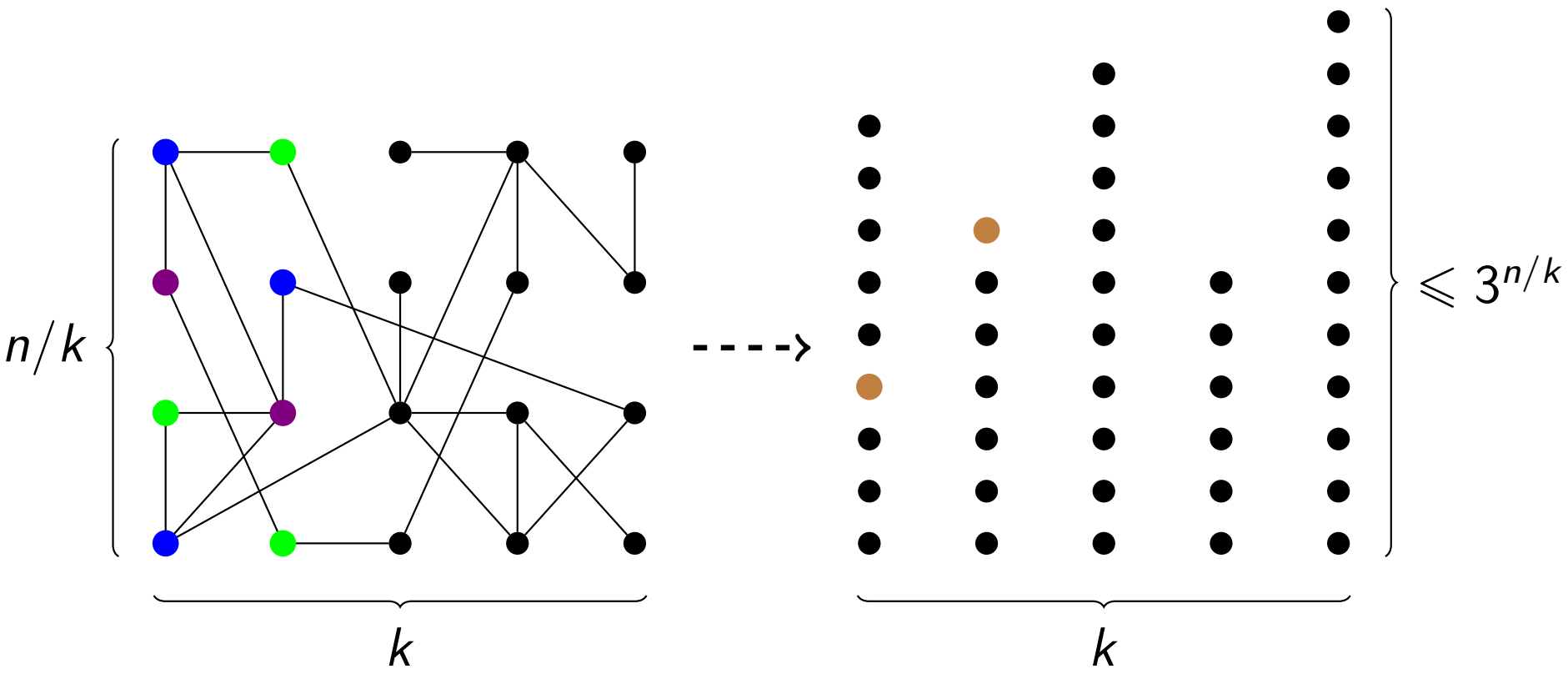
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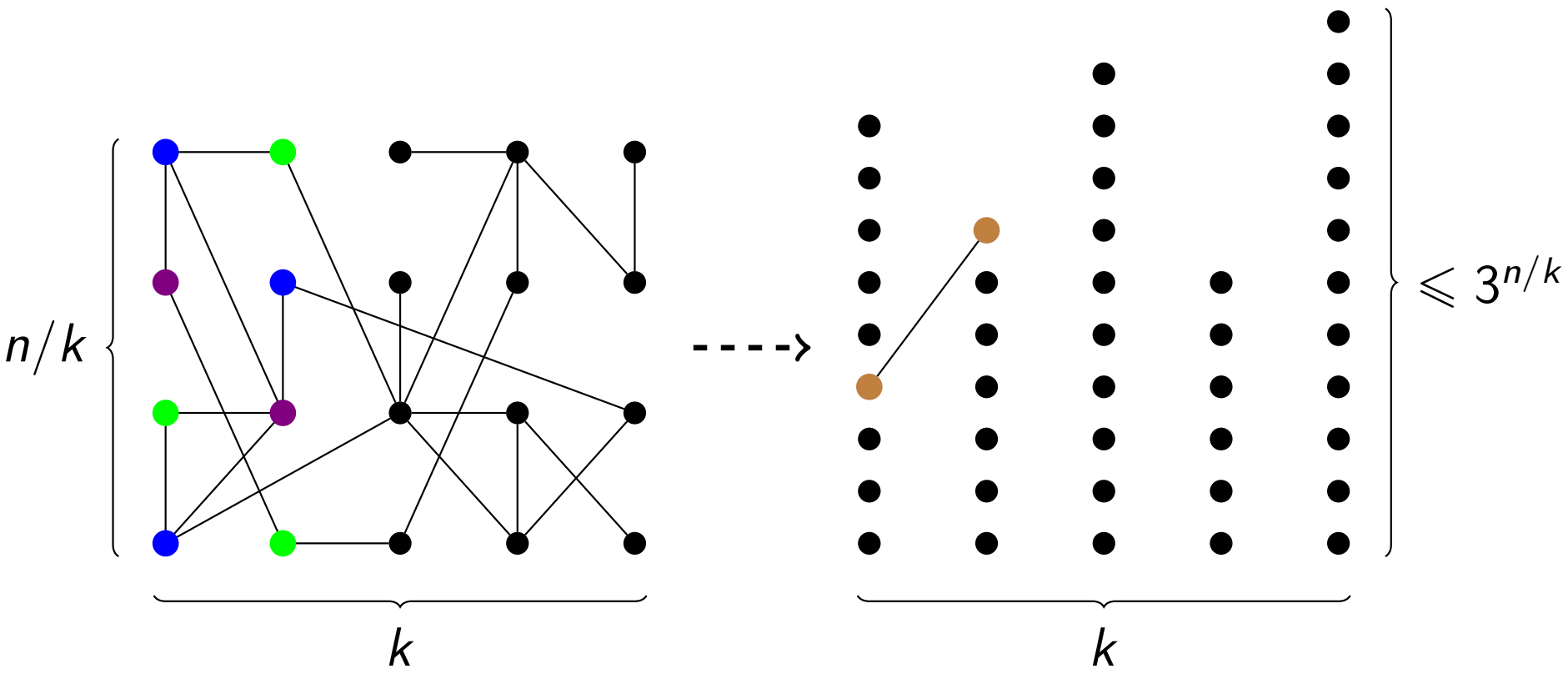
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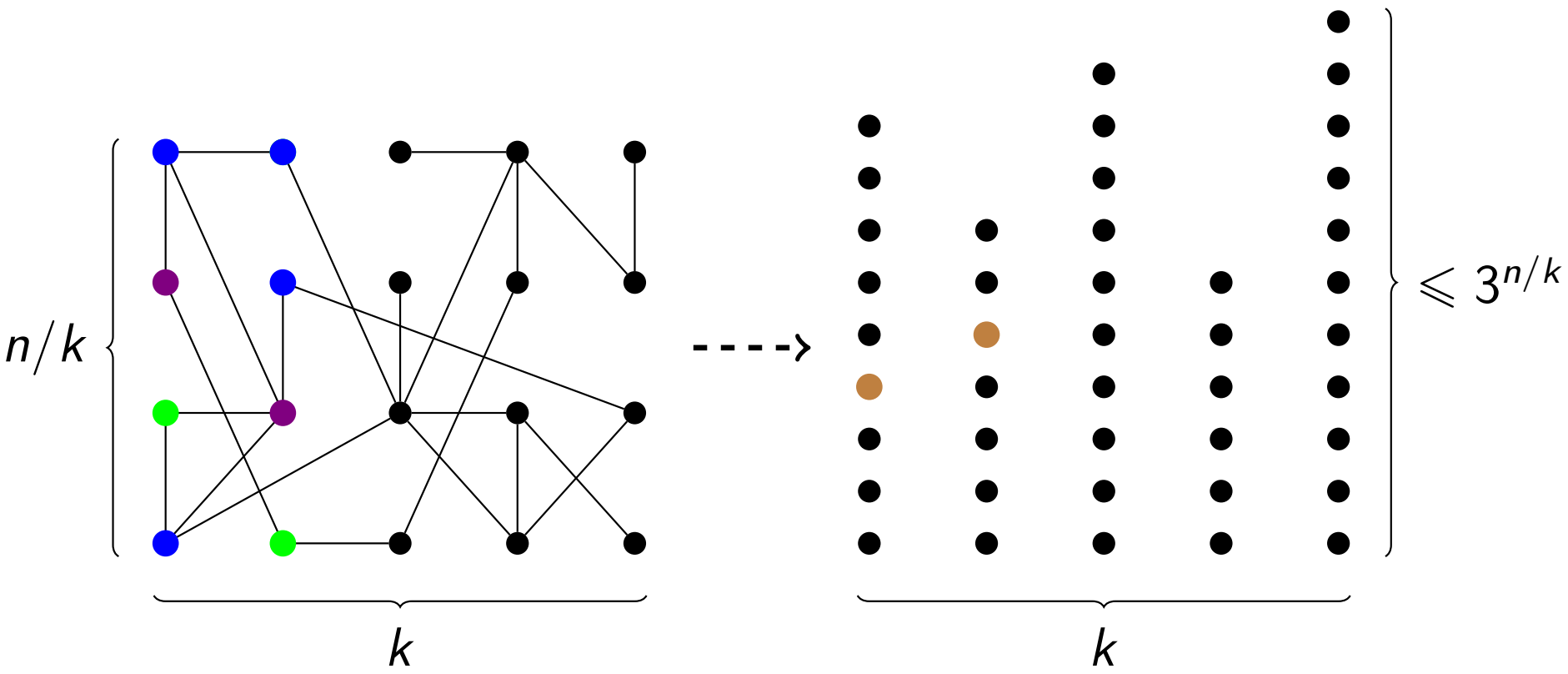
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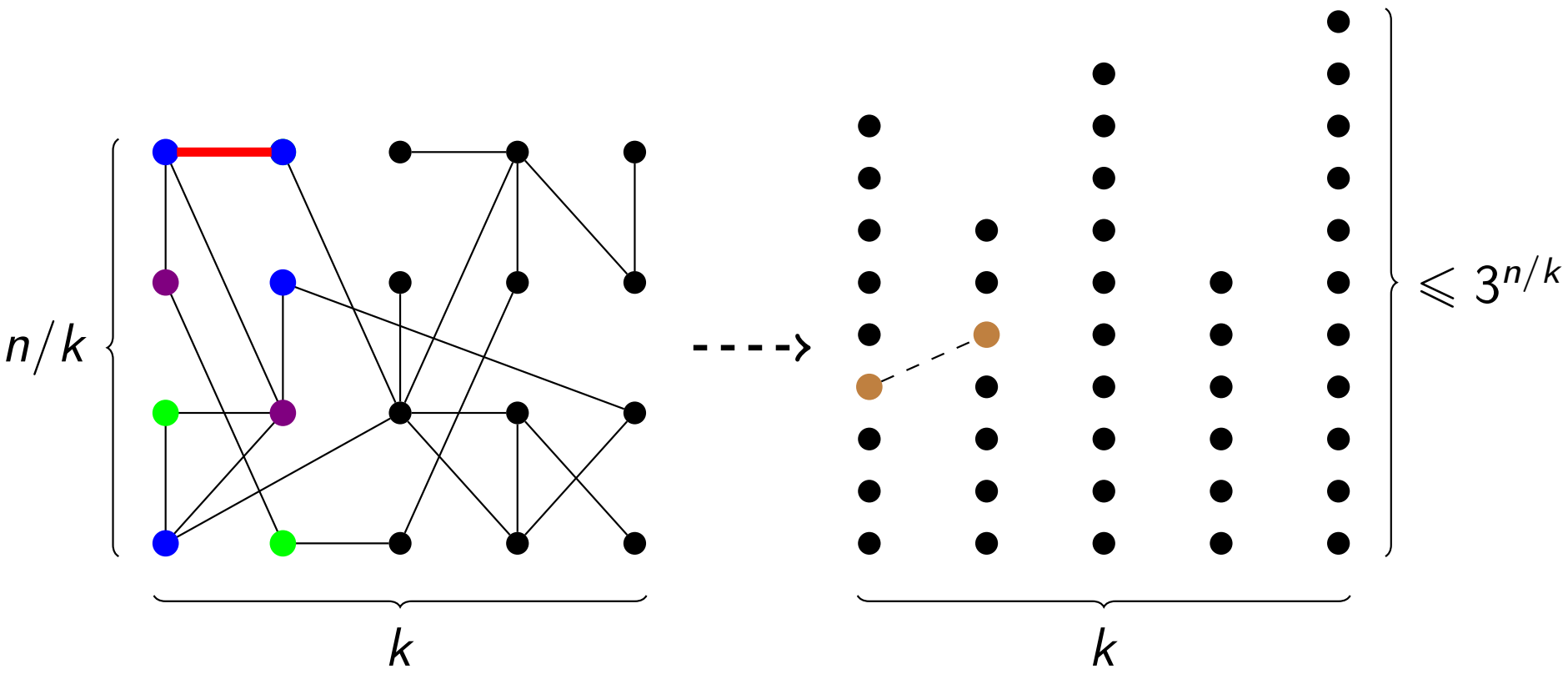
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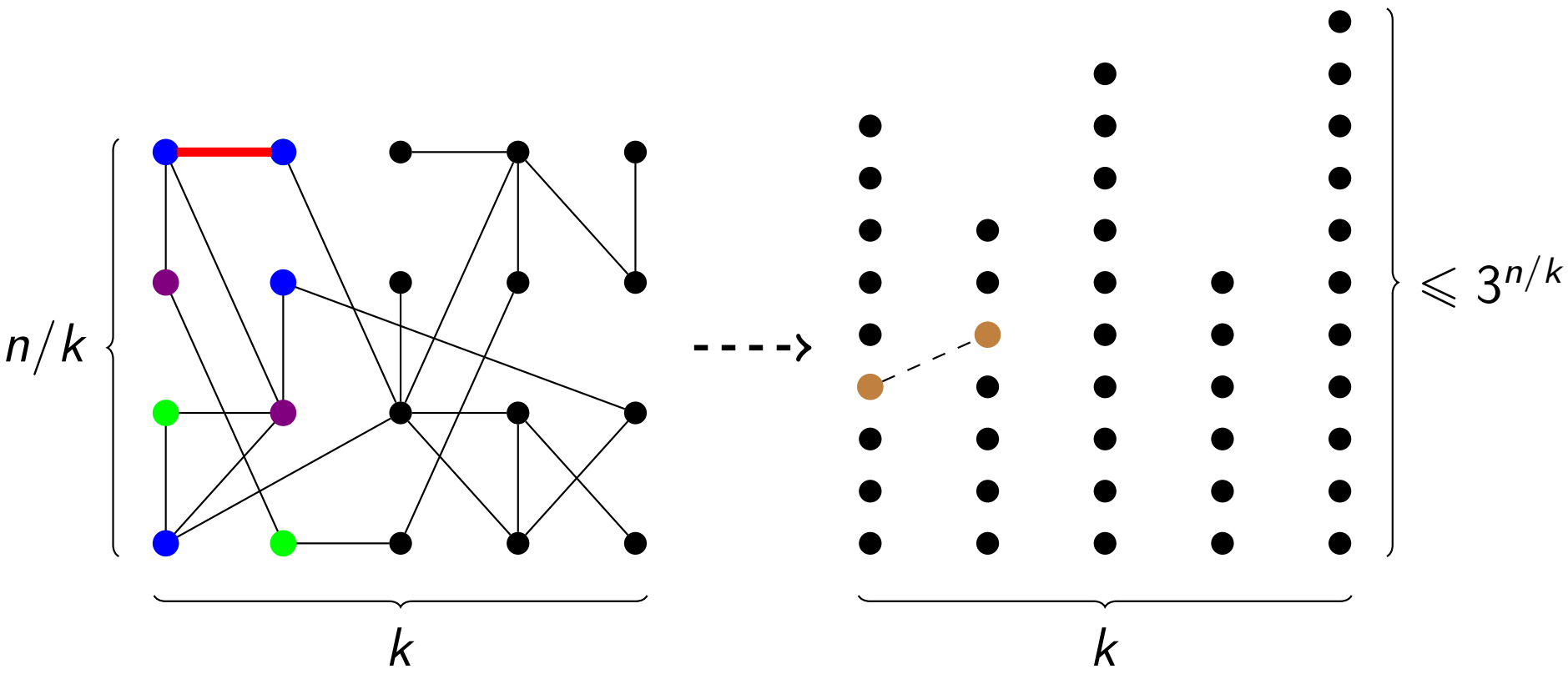

Left graph admits 3-coloring iff right graph contains k-clique.

Theorem (implies W[1] $\neq$ FPT)

Unless ETH fails, there is no $f(k)n^{o(k)}$ time algorithm for Clique.

Proof continued:

- Created graph has $N = k \cdot 3^{n/k}$ vertices.

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- To exclude all computable $f(k)$, one needs roughly $f^{-1}(n)$ (technical difficulties omitted).

Theorem (implies W[1] $\neq$ FPT)

Unless ETH fails, there is no $f(k)n^{o(k)}$ time algorithm for Clique.

- A reduction from Clique to X which outputs parameter $g(k)$ implies lower bound of $f(k)n^{o(g^{-1}(k))}$.

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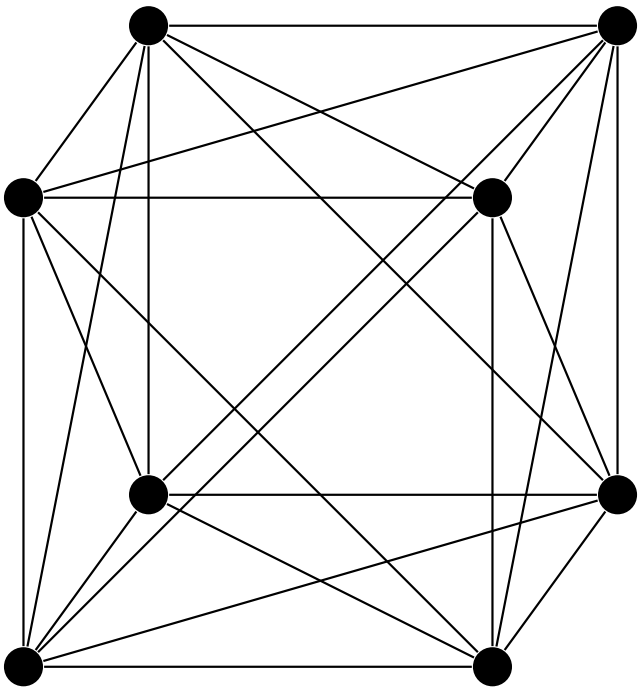
- Consequently (under ETH) there is no $f(k)n^{o(k)}$ algorithm for those problems.
- For Odd Set the reduction does not give so strong lower bound (Ex: check what is implied by the reduction from lecture 4).

ETH used for somewhat exotic lower bounds

Edge Clique Cover (ECC)

Input: Graph G, integer k

Question: Does there exists a collection of k cliques $C_1, C_2, \ldots, C_k$ in G, such that $E(G) = \bigcup_{i=1}^{k} E(C_i)$?



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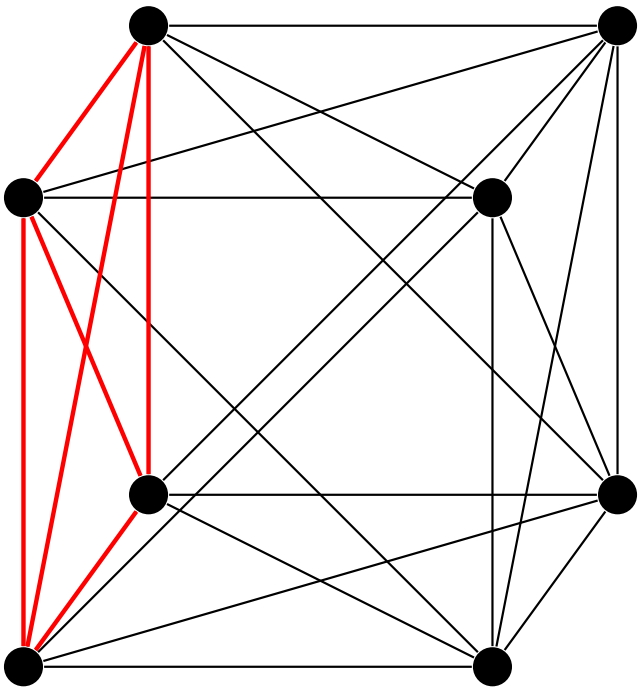

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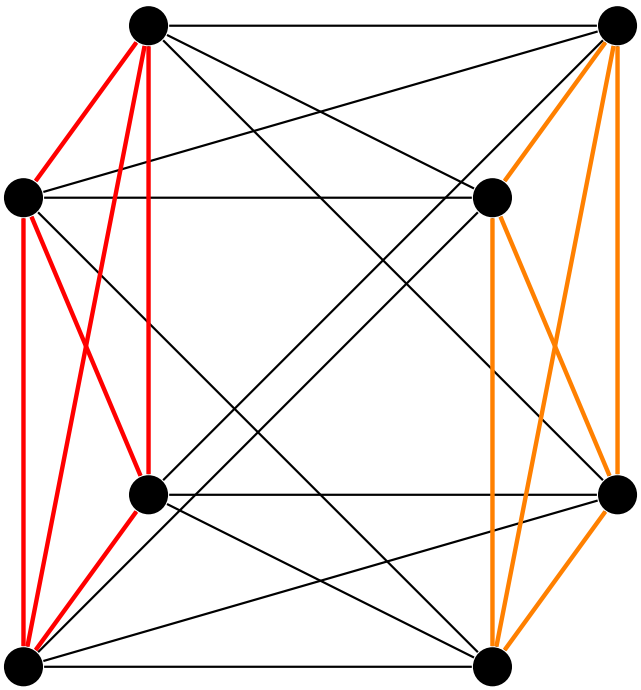

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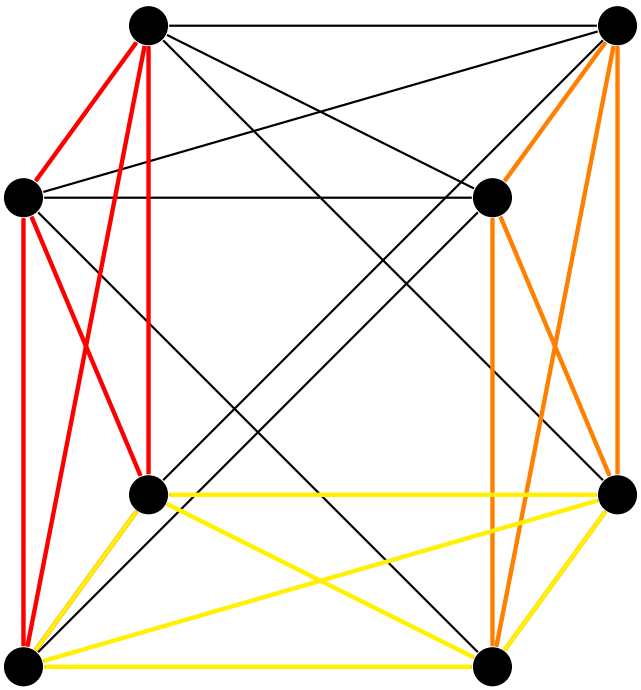

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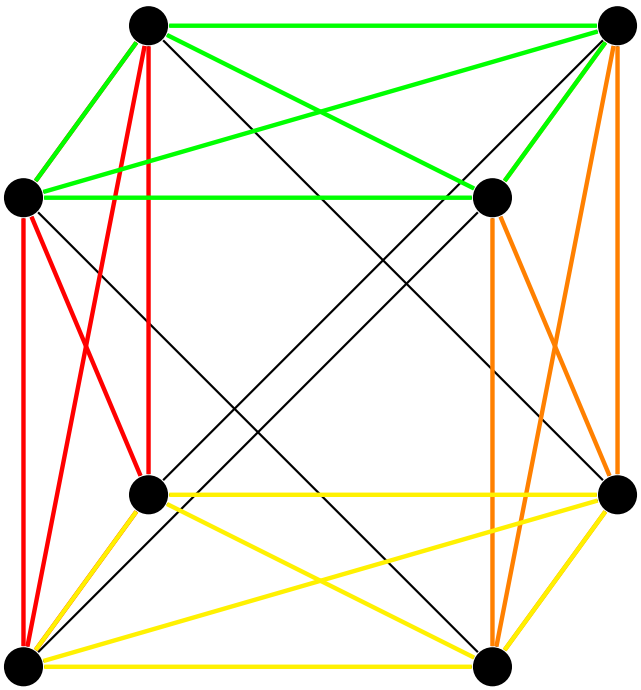

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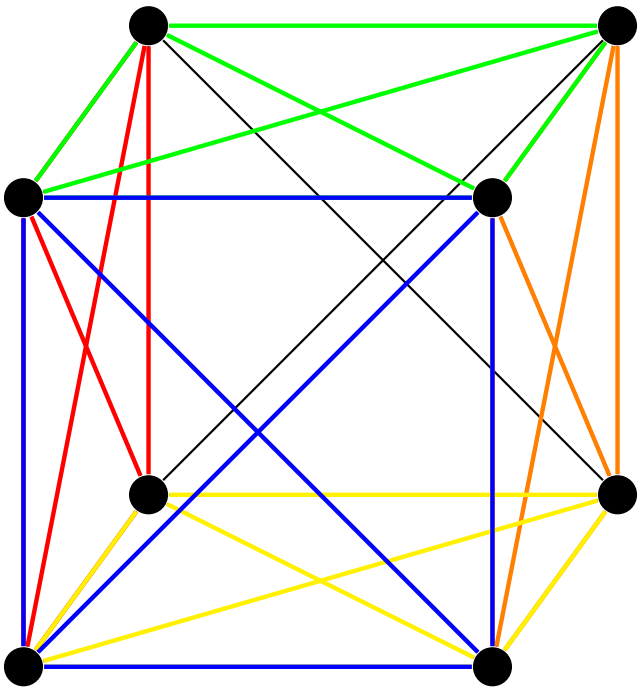

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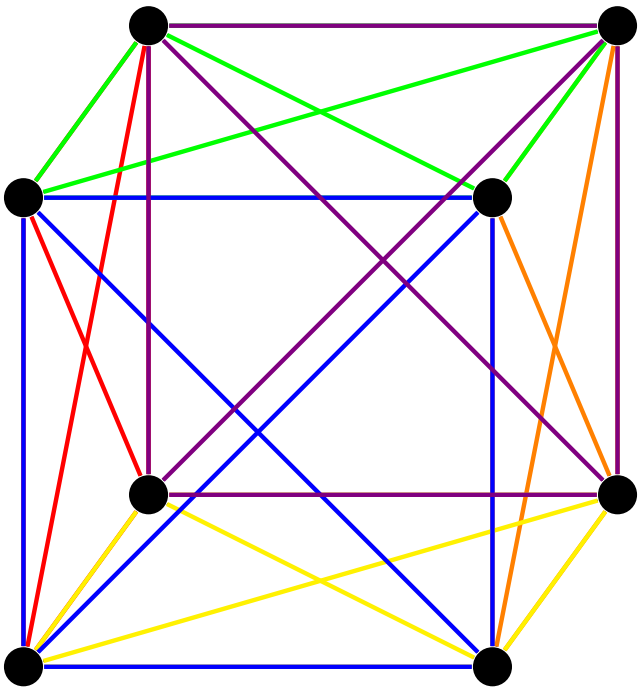

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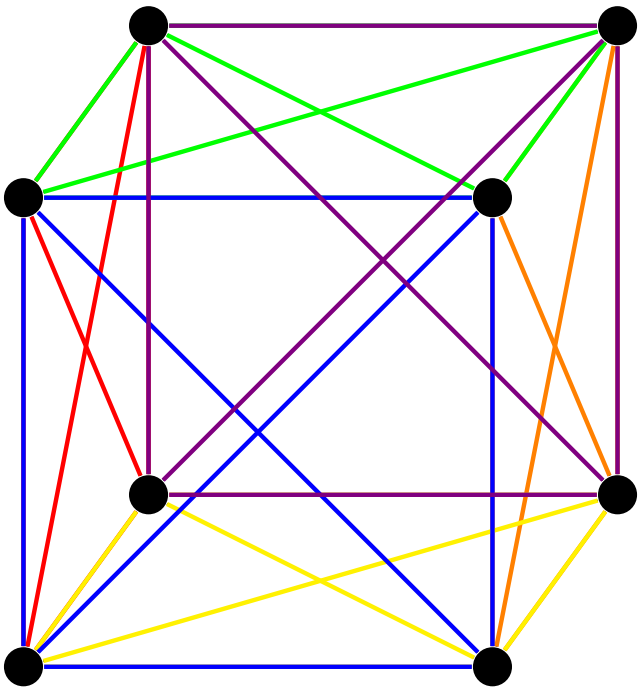

6 cliques, but not optimal

puzzle: find a solution with 5 cliques

ECC

Book, section 2.3.3

Edge Clique Cover admits a kernel with at most 2^k vertices.

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Exercise

By casting the problem as Set Cover, we obtain $\mathcal{O}^*(2^{2^{\mathcal{O}(k)}})$ algorithm.

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Exercise

By casting the problem as Set Cover, we obtain $\mathcal{O}^*(2^{2^{\mathcal{O}(k)}})$ algorithm.

Theorem

Unless ETH fails, there is no $\mathcal{O}^*(2^{2^{o(k)}})$ time algorithm for ECC.
Proof by reducing formula with n vars, m clauses to poly size ECC instance with $k = \mathcal{O}(\log(n+m))$.

Closest Substring

Closest Substring

Input: strings $s_1, \ldots, s_t \in \Sigma^*$, integers L, d.
Question: Is there $s \in \Sigma^L$, such that for every $1 \leqslant i \leqslant t$ there is a substring s_i' of s_i of length L with $d_H(s, s_i) \leqslant d$.

Closest Substring

Closest Substring - important parts

Input: t strings of total size n

Question: fairly natural question about Hamming distance

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Lemma

Closest Substring over constant size alphabet can be solved in time $f(d,t)n^{\mathcal{O}(\log\log t)}$. Mover under ETH there is no $f(d,t)n^{o(\log\log t)}$ time algorithm for any f.

Summary

Take-home message 1

$$\text{SETH} \implies \text{ETH} \implies \text{W[1]} \neq \text{FPT} \implies \text{P} \neq \text{NP}$$

$$\text{FPT} \neq \text{W[1]} \implies \text{no } \mathcal{O}^*(f(k))$$
$$\text{ETH} \implies \text{no } \mathcal{O}^*(2^{o(f(k))})$$
$$\text{SETH} \implies \text{no } \mathcal{O}^*(2^{(1-\epsilon)f(k)}).$$

Take-home message 2

When showing lower bounds, we have to come up with a reduction and track what happens with the parameter in the new instance.

Summary

- We have learned that ETH is a tool that allows proving exciting lower bounds.
- More on this subject on Thursday (lectures by Michał and Daniel L.).
- Exercises: 14.1, 14.2.(d) (+OCT), show $\mathcal{O}^*(2^{2^{O(k)}})$ algorithm for ECC (after kernelization), check what lower bound (under ETH) is implied for Odd Set by reduction from lecture 4.

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Thank you for your attention

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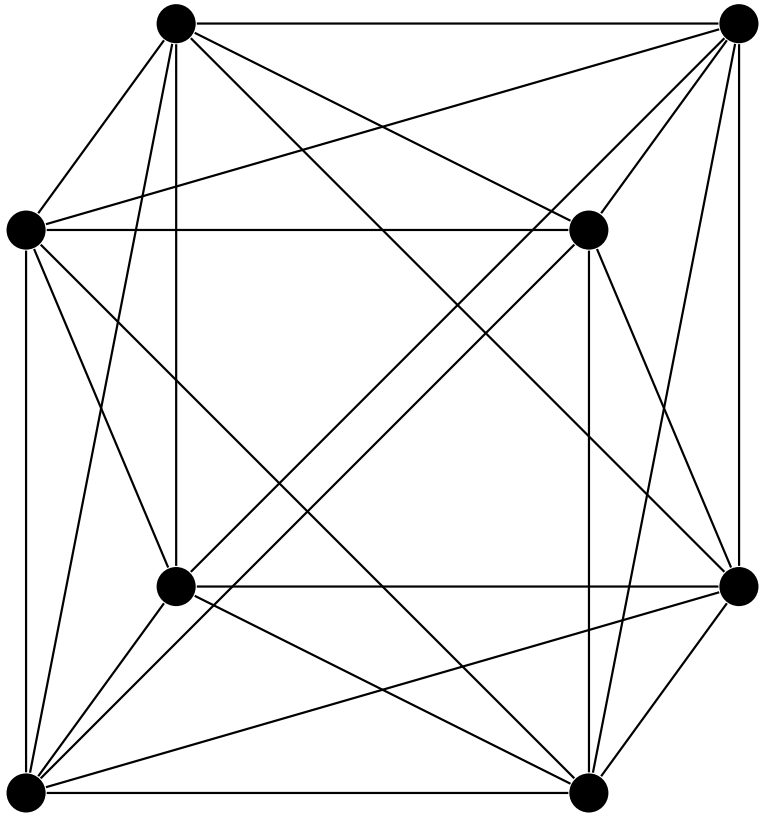

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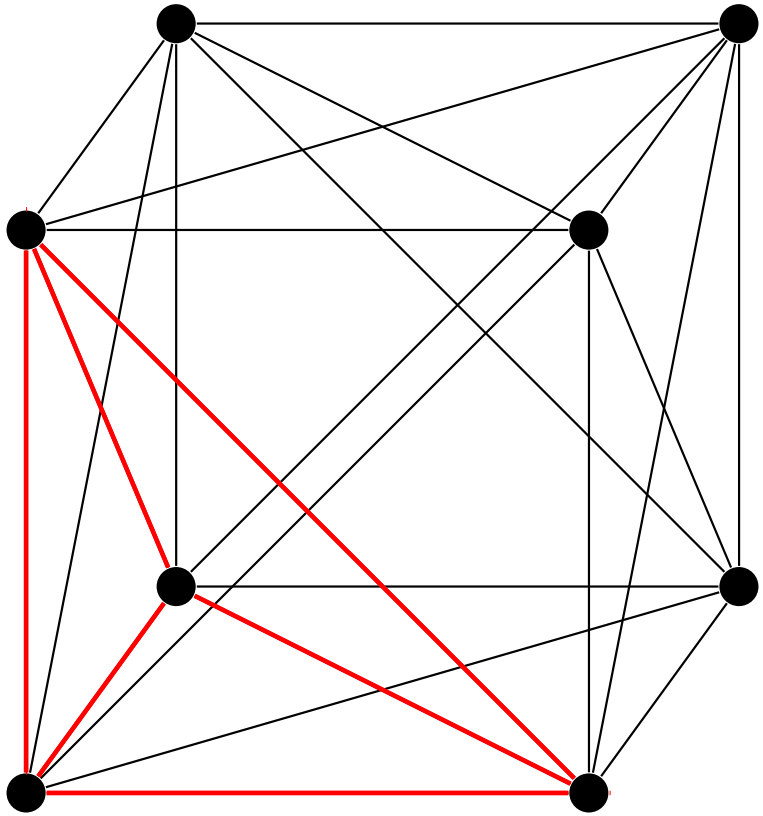

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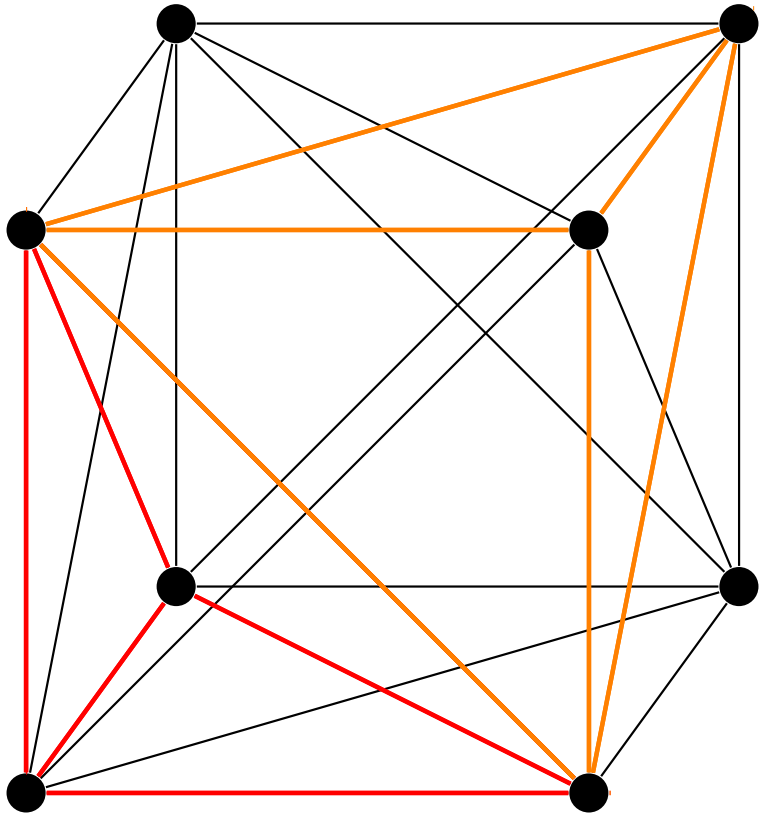

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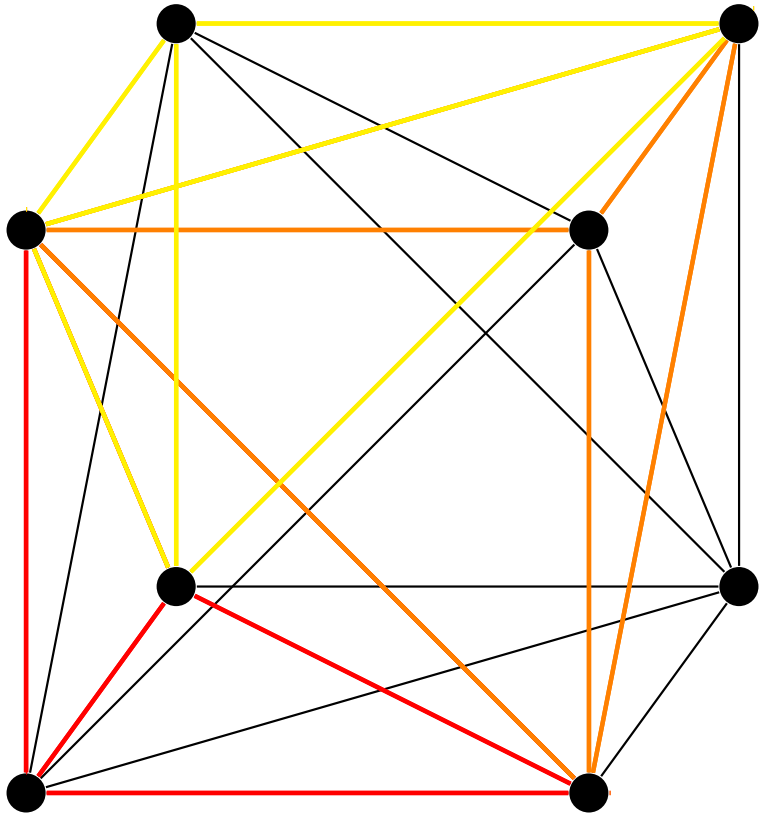

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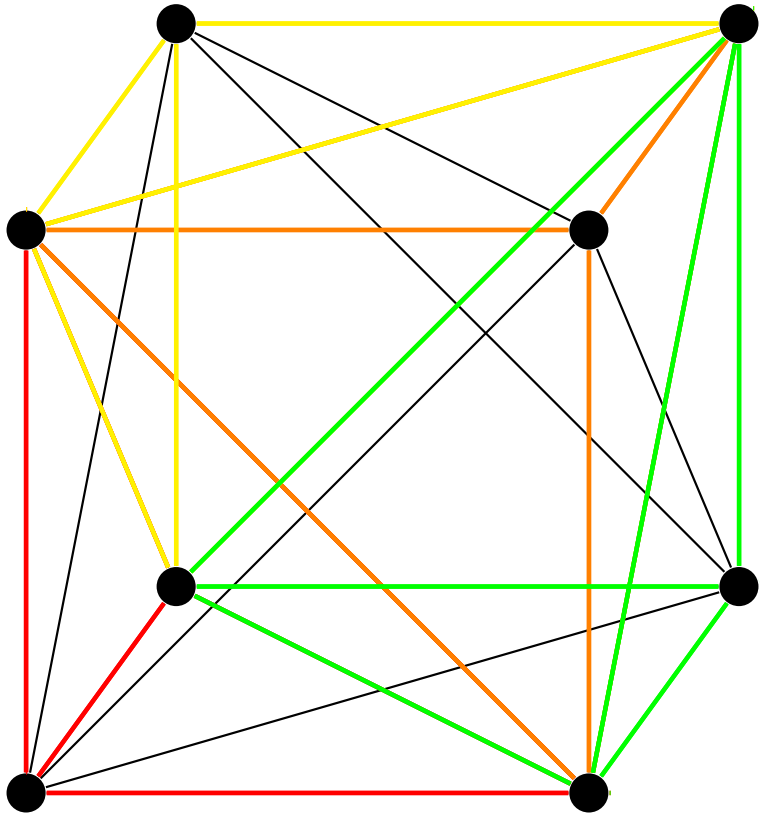

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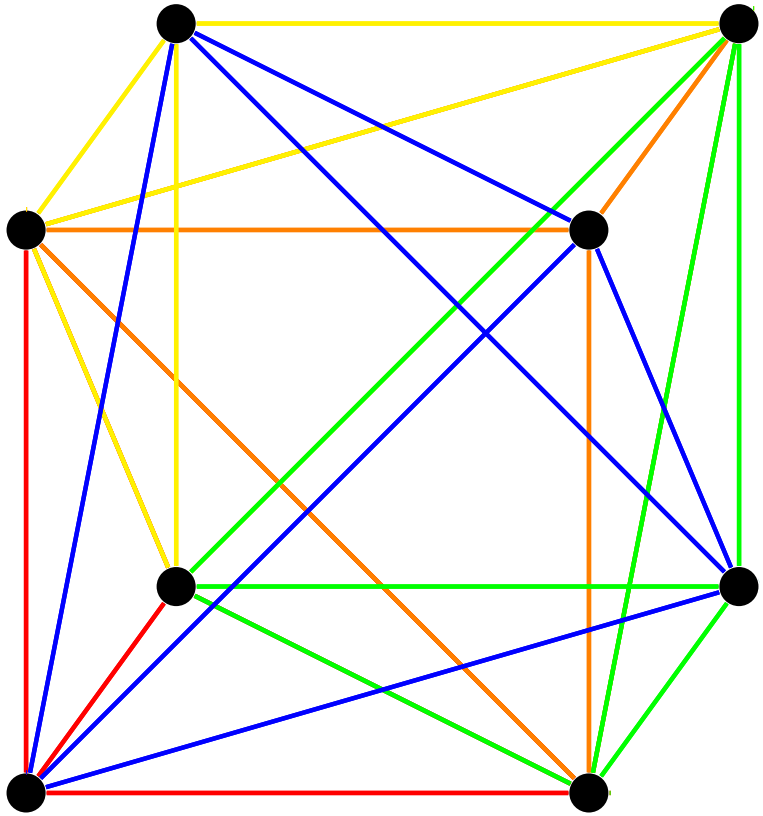

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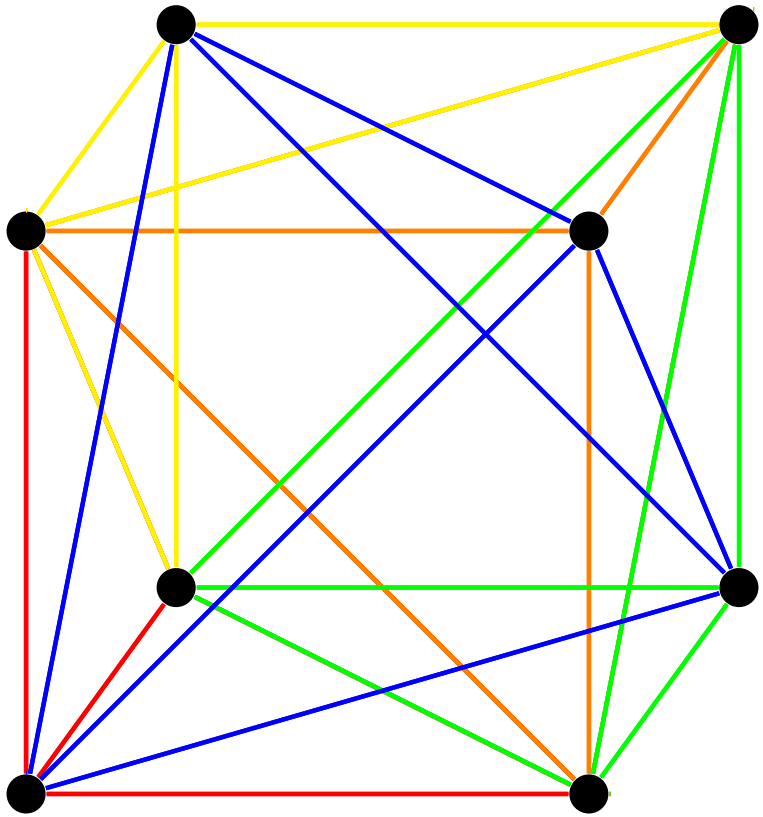

Magical solution with 5 cliques