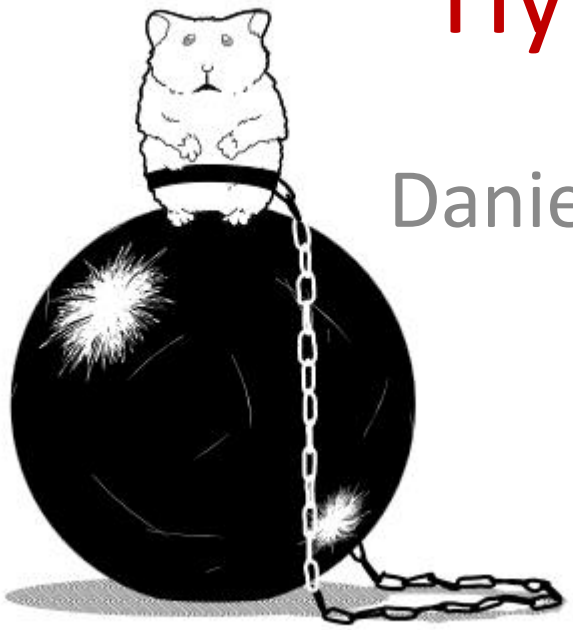
The Strong Exponential Time Hypothesis



Daniel Lokshtnaov

Tight lower bounds

Have seen that ETH can give tight lower bounds

How tight? ETH «ignores» constants in exponent

How to distinguish 1.85^n from 1.0001^n?

SAT

Input: Formula ψ with m clauses over n boolean variables.

Question: Does there exist an assignment to the variables that satisfies all clauses?

Note: Input can have size superpolynomial in n!

Fastest algorithm for SAT: $2^n \text{poly}(m)$

d-SAT

Here all clauses have size $\leq$ d

Input size $\leq n^d$

Fastest algorithm for 2-SAT:	n+m
Fastest algorithm for 3-SAT:	1.31^n
Fastest algorithm for 4-SAT:	1.47^n
…	
Fastest algorithm for d-SAT:	$2^{(1-c/d)n}$
Fastest algorithm for SAT:	2^n

Strong ETH

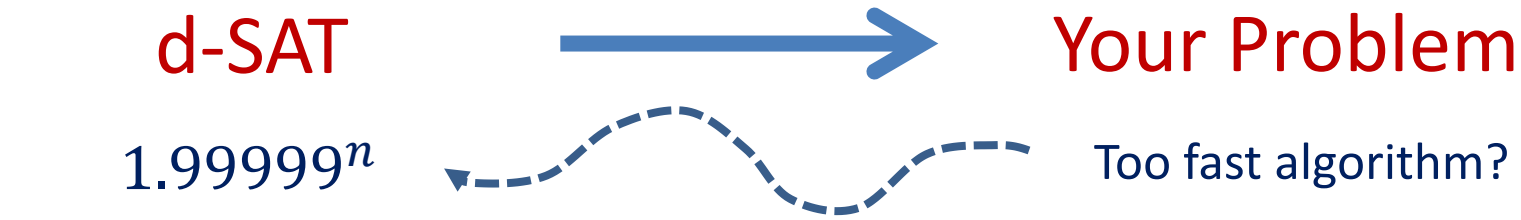
Let $s_d = \inf\{c : \text{d-SAT has a } 2^{cn} \text{ algorithm}\}$

Let $s_\infty = \lim_{d\to\infty} s_d$

Know: $0 \le s_d \le s_\infty \le 1$

ETH: $s_3 > 0$ SETH: $s_\infty = 1$

Showing Lower Bounds under SETH



The number of 9's MUST
be independent of d

Dominating Set

Input: n vertices, integer k

Question: Is there a set S of at most k vertices such that $N[S] = V(G)$?

Naive: n^{k+1}

Smarter: $n^{k+o(1)}$

Assuming ETH: no $f(k)n^{o(k)}$

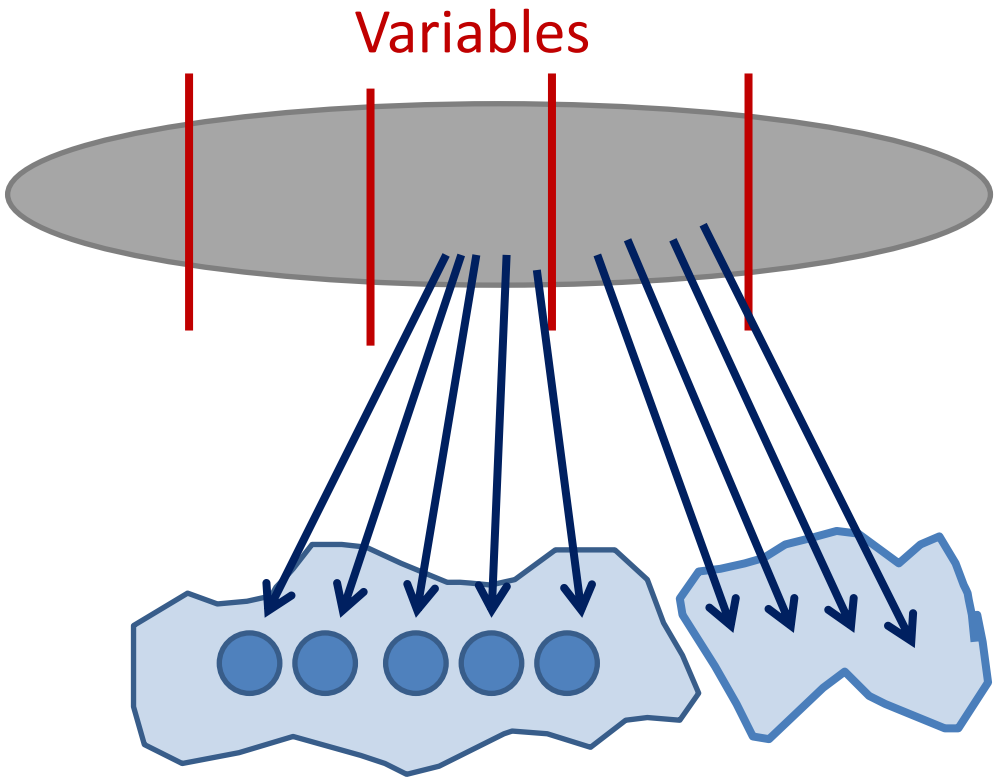
$n^{k/10}$?

n^{k-1}?

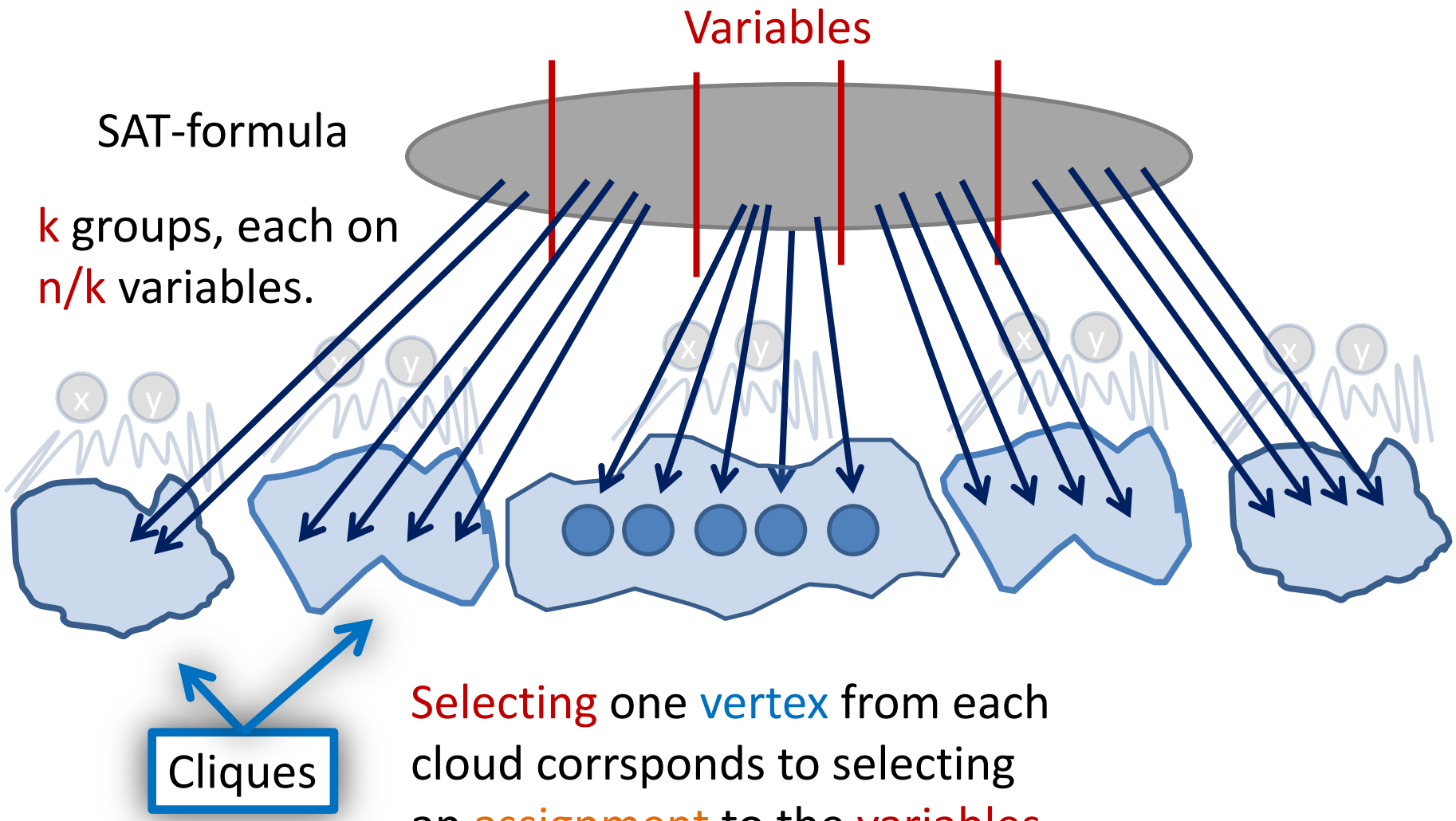
SAT → k-Dominating Set

SAT-formula

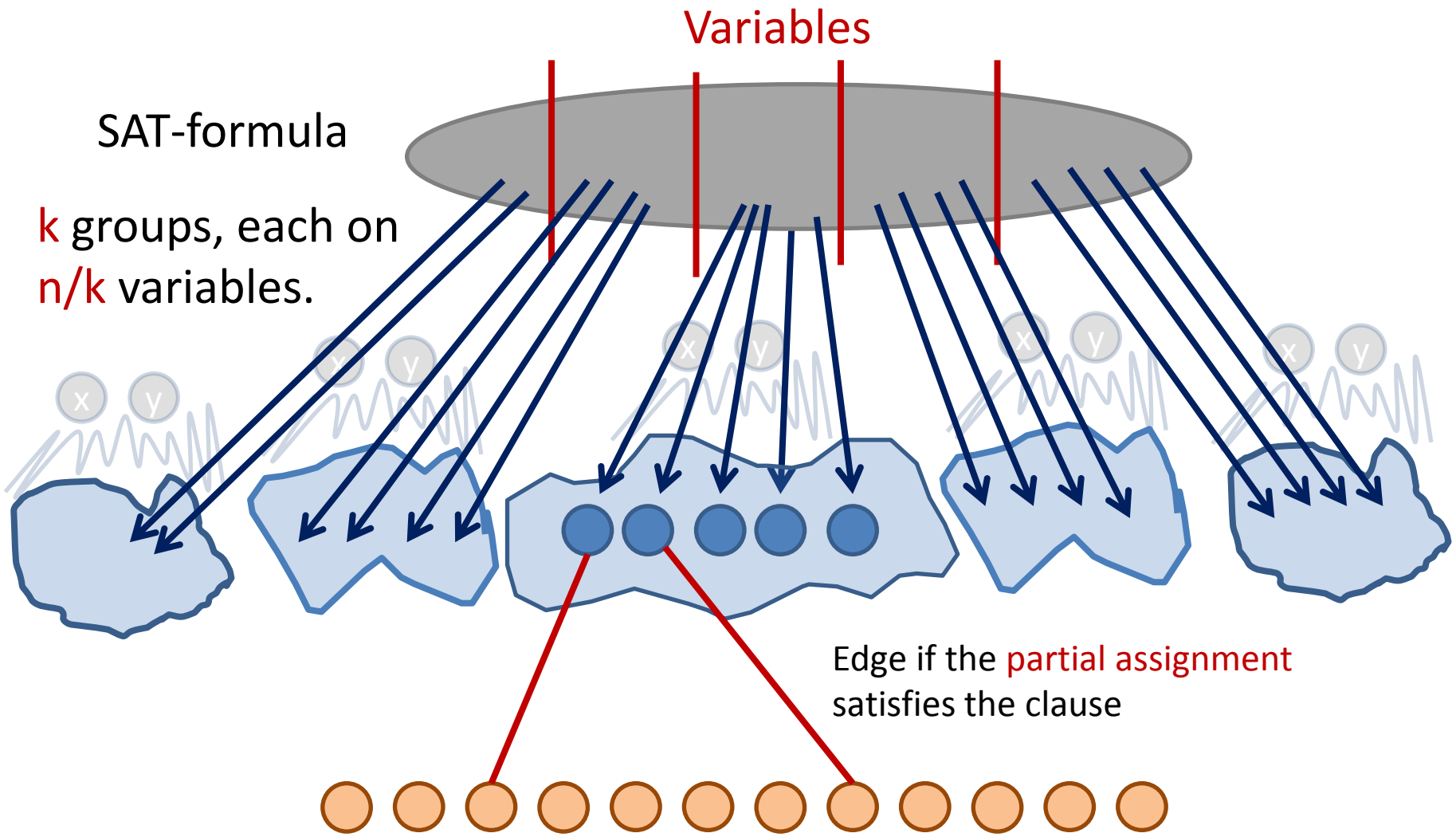
k groups, each on n/k variables.



One vertex for each of the $2^{n/k}$ assignments to the variables in the group.



Selecting one vertex from each cloud corrsponds to selecting an assignment to the variables.


Variables
SAT-formula
k groups, each on
n/k variables.
x
y
Edge if the partial assignment
satisfies the clause
One vertex per clause in the formula

SAT → k-Dominating Set

analysis

Too fast algorithm for k-Dominating Set: $n^{k-0.01}$

For any fixed k (like k=3)

If $m \geq 2^{n/k}$ then 2^n is at most m^k,
which is polynomial!
So $m \leq 2^{n/k}$

The output graph has
$k2^{n/k} + m \leq 2k \cdot 2^{n/k}$ vertices

$$(2k \cdot 2^{n/k})^{k-0.01} \leq (2k)^k \cdot 2^{n\frac{k-0.01}{k}}$$

$$= O(1.999^n)$$

Dominating Set, wrapping up

A $O(n^{2.99})$ algorithm for 3-Dominating Set, or

a $O(n^{3.99})$ algorithm for 4-Dominating Set, or a

a $O(n^{4.99})$ algorithm for 5-Dominating Set, or a …

… would violate SETH.

Independent Set / Treewidth

Input: Graph G, integer k, tree-decomposition of G of width $\leq t$.

Question: Does G have an independent set of size at least k?

DP: $O(2^t n)$ time algorithm

Can we do it in 1.99^t poly(n) time?

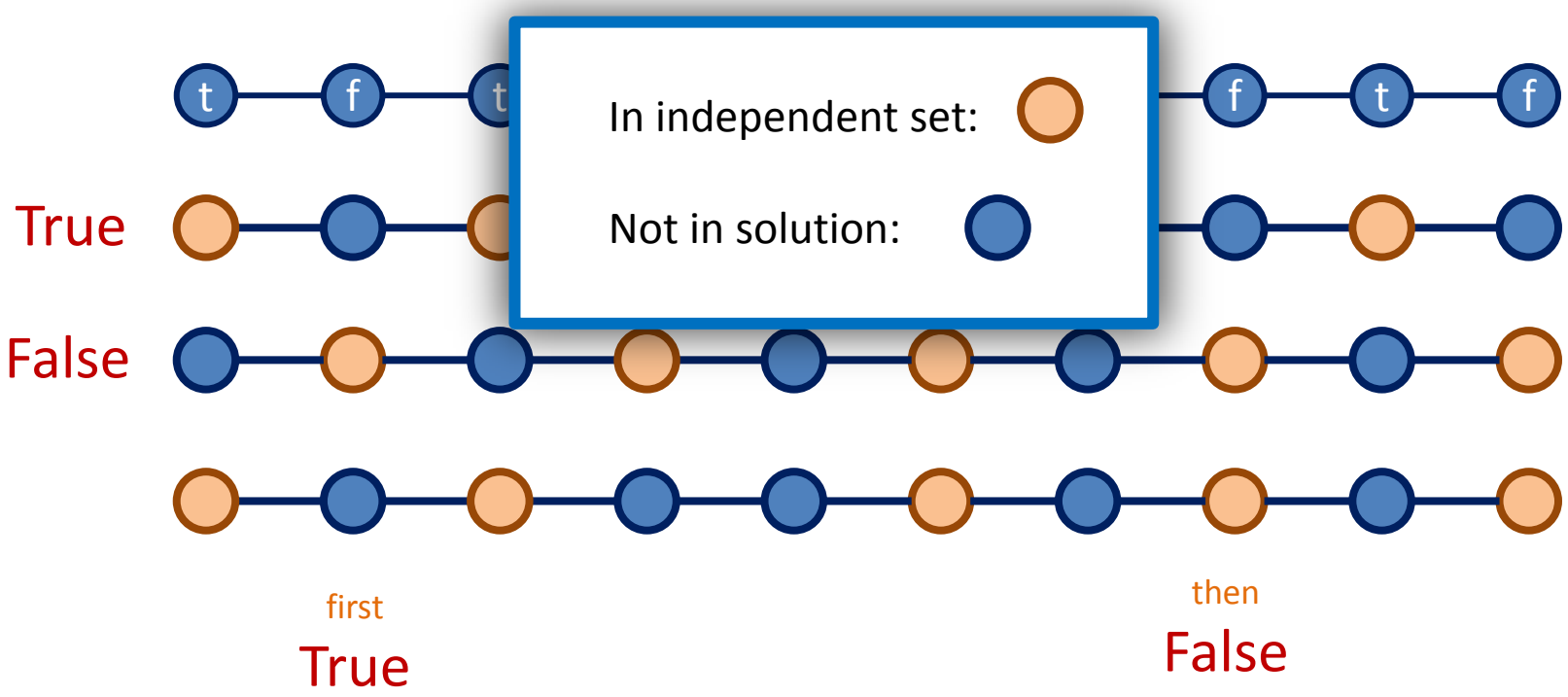
Next: If yes, then SETH fails!

Independent Set / Treewidth

Will reduce n-variable d-SAT to Independent Set in graphs of treewidth t, where $t \leq n+d$.

So a $1.99^t poly(N)$ algorithm for Independent Set gives a $1.99^{n+d} poly(n) \leq O(1.999^n)$ time algorithm for d-SAT.

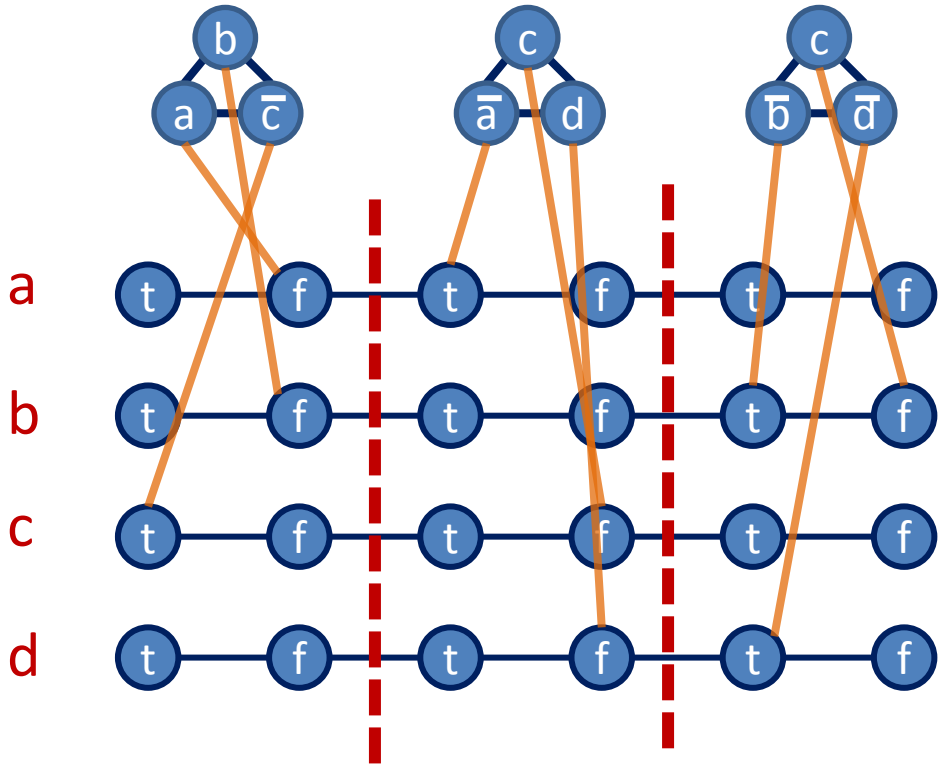
Independent Sets on an Even Path



d-SAT $\leq$ Independent Set

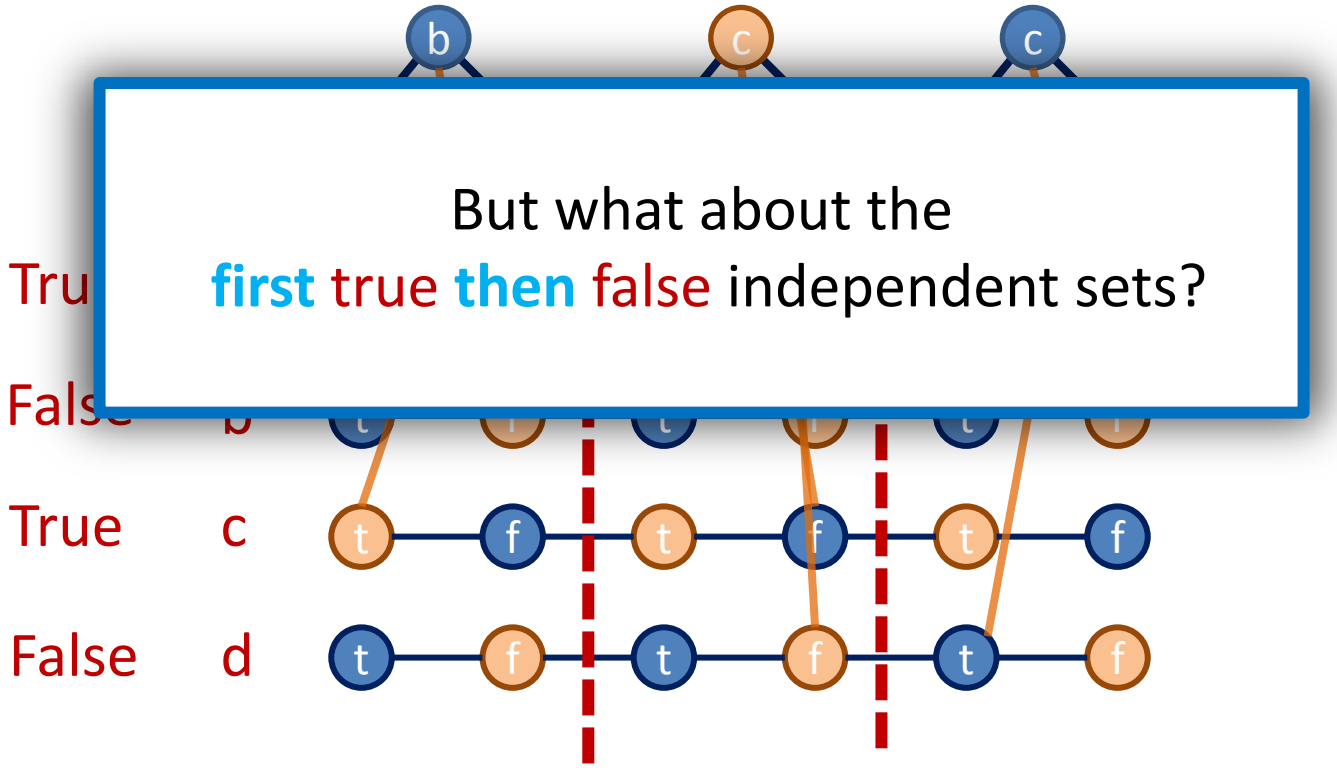
proof by example

$\psi = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee d) \wedge (\bar{b} \vee c \vee \bar{d})$

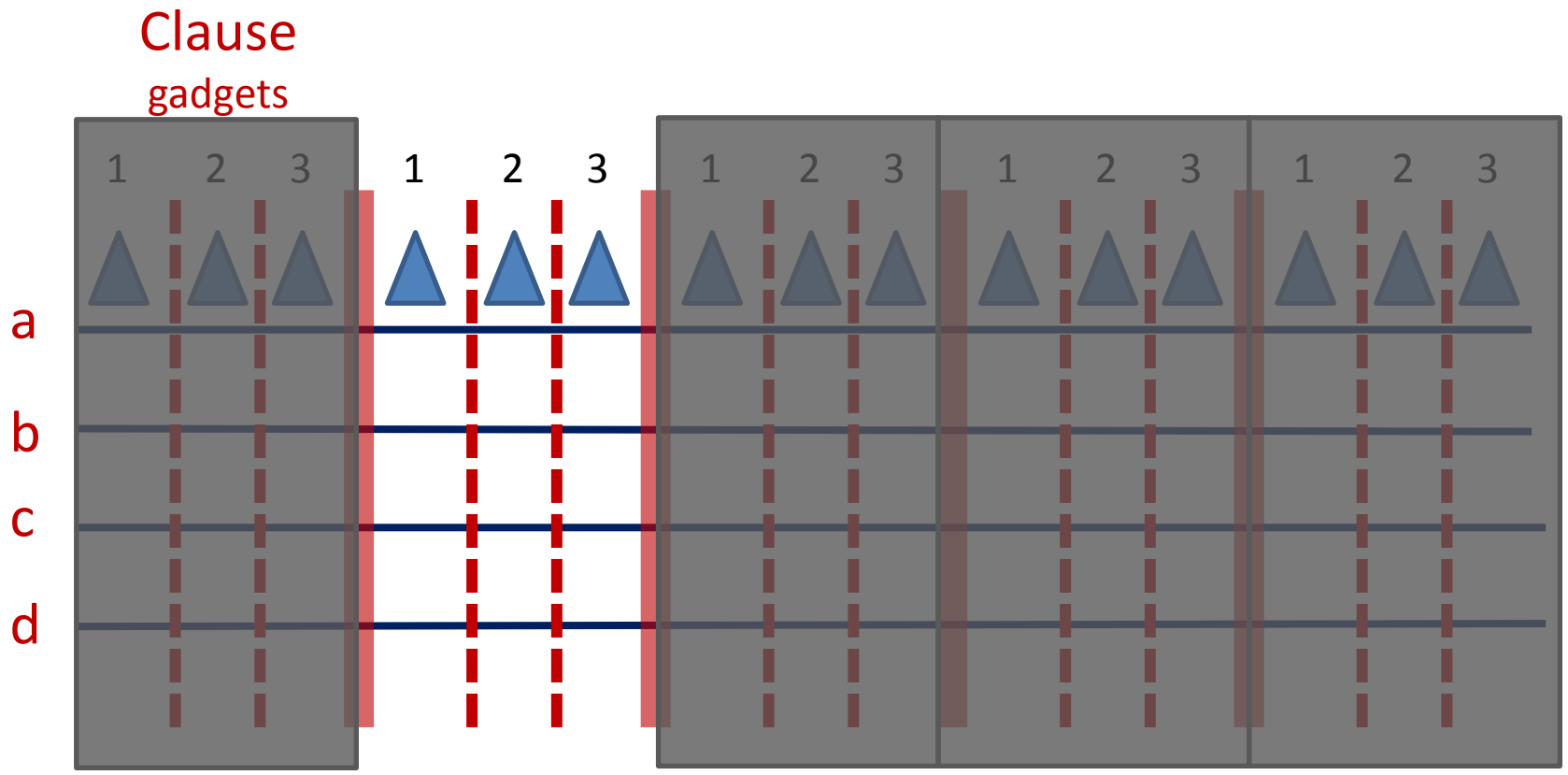


Independent Sets ↔ Assignments

$$\psi = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee d) \wedge (\bar{b} \vee c \vee \bar{d})$$



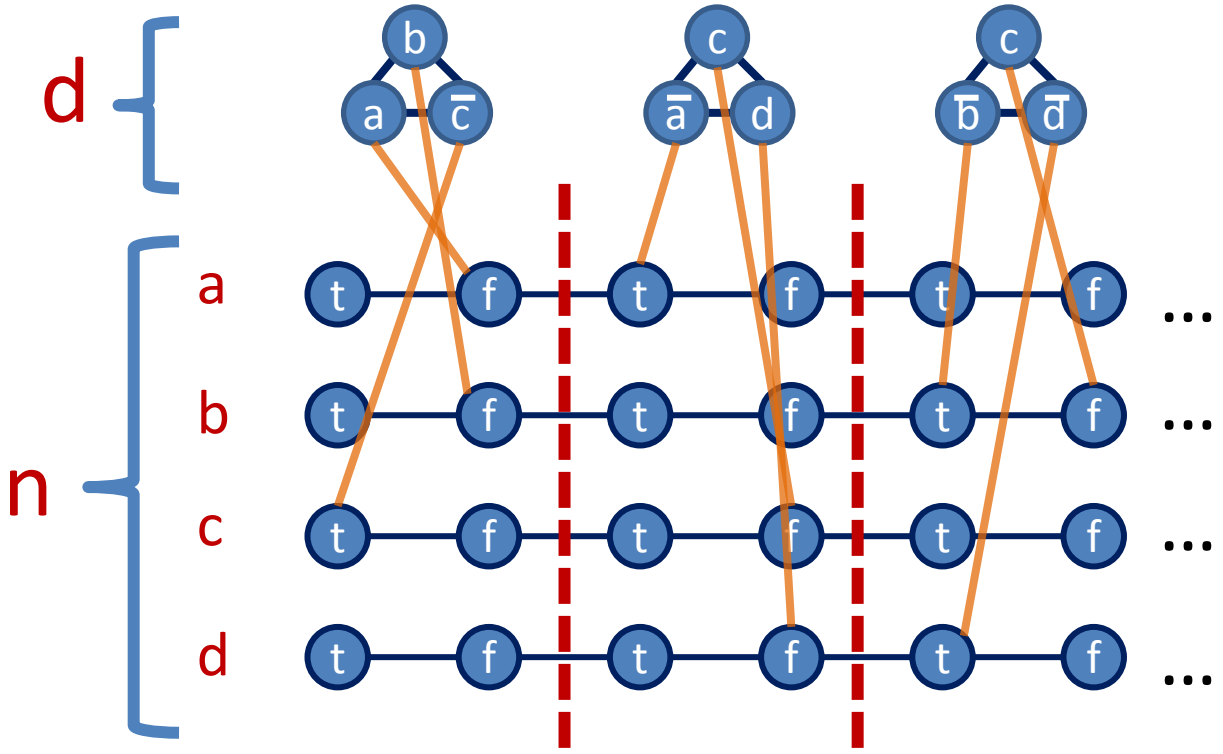
Dealing with true→false



Every variable flips true→false at most once!

Treewidth Bound

by picture



Formal proof - exercise

Independent Set / Treewidth

wrap up

Reduced n-variable d-SAT to Independent Set in graphs of treewidth t, where $t \leq n+d$.

A $1.99^t poly(N)$ algorithm for Independent Set gives a $1.99^{n+d} poly(n) \leq O(1.999^n)$ time algorithm for d-SAT.

Thus, no 1.99^t algorithm for Independent Set assuming SETH

Assuming SETH, the following algorithms are optimal:

- $2^t \cdot \text{poly}(n)$ for Independent Set
- $3^t \cdot \text{poly}(n)$ for Dominating Set
- $c^t \cdot \text{poly}(n)$ for c-Coloring
- $3^t \cdot \text{poly}(n)$ for Odd Cycle Transversal
- $2^t \cdot \text{poly}(n)$ for Partition Into Triangles
- $2^t \cdot \text{poly}(n)$ for Max Cut
- …

3^t lower bound for Dominating Set?

Need to reduce k-SAT formulas on n-variables to Dominating Set in graphs of treewidth t, where

$$3^t \approx 2^n$$

So $t \approx {}^{n}/_{\log 3}$

Hitting Set / n

Input: Family $F = \{S_1,\ldots,S_m\}$ of sets over universe $U = \{v_1, \ldots, v_n\}$, integer k.

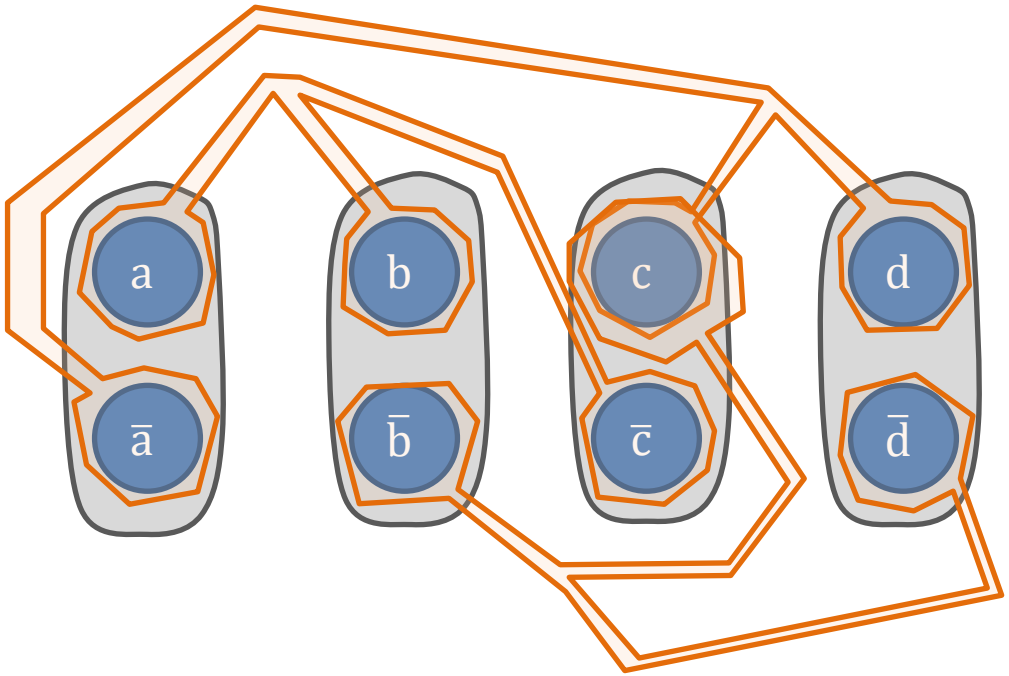
Question: Does there exist a set $X \subseteq U$ of size at most k such that for every $S_i \in F$, $S_i \cap X \neq \emptyset$?

Naive algorithm runs in $O(2^n nm)$ time.

Next: $1.41^n \mathrm{poly}(n, m)$ implies that SETH fails

d-SAT $\leq$ Hitting Set

$\psi = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee d) \wedge (\bar{b} \vee c \vee \bar{d})$



Budget = 4

d-SAT vs Hitting Set

A c^n algorithm for Hitting Set makes a c^{2n} algorithm for d-SAT.

Since $1.41^{2n} < 1.9999^n$, a 1.41^n algorithm for Hitting Set violates the SETH.

Have a 2^n algorithm and a 1.41^n lower bound.

Next: 2^n lower bound

Hitting Set

For any fixed $\epsilon > 0$, will reduce k-SAT with n variables to Hitting Set with universe with at most $(1+\epsilon)n$ elements.

So a 1.99^n algorithm for Hitting Set gives a $1.99^{n(1+\epsilon)} \leq 1.999^n$ time algorithm for k-SAT

Some deep math

For every $\epsilon > 0$ there exists a natural number g such that, for $t = \lfloor g(1+\epsilon) \rfloor_{\text{odd}}$ we have:

$$\binom{t}{\lceil t/2 \rceil} \geq 2^g$$

Why is this relevant?

d-SAT ≤ Hitting Set

Group the variables into groups of size g, and set

t = [illegible]

Solution budget $\lceil t/2 \rceil$ from each group

Will force $\geq \lceil t/2 \rceil$ from each group

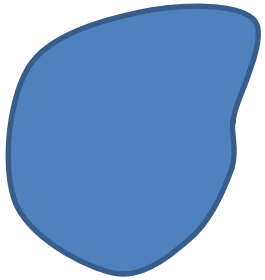
→ Exactly $\lceil t/2 \rceil$ from each group

Element

Elements ≤ (1 + ϵ) · variables

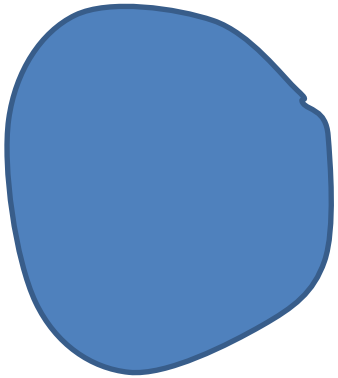
Analyzing a group

Group of g variables



2^g assignments to variables

Injection



$\binom{t}{\lceil t/2 \rceil}$ subsets of elements of size exactly $\lceil t/2 \rceil$.

Group of t elements

Forcing solution $\lceil t/2 \rceil$ vertices in a group?

Add all subsets of the group of size $\lceil t/2 \rceil$ to the family F.

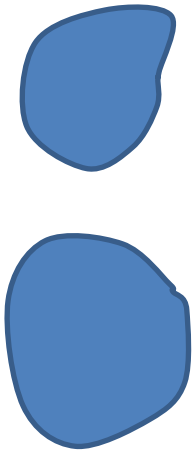
Lets call these sets guards

Any set that picks less than $\lceil t/2 \rceil$ elements the group misses a guard.

Any set that picks at least $\lceil t/2 \rceil$ elements from each group hits all the guards

Analyzing a group

Group of g variables



assignments to variables

subsets of elements of size exactly $\lceil t/2 \rceil$.

Group of t elements

What about the element subsets of size $\lceil t/2 \rceil$ that do not correspond to assignments?

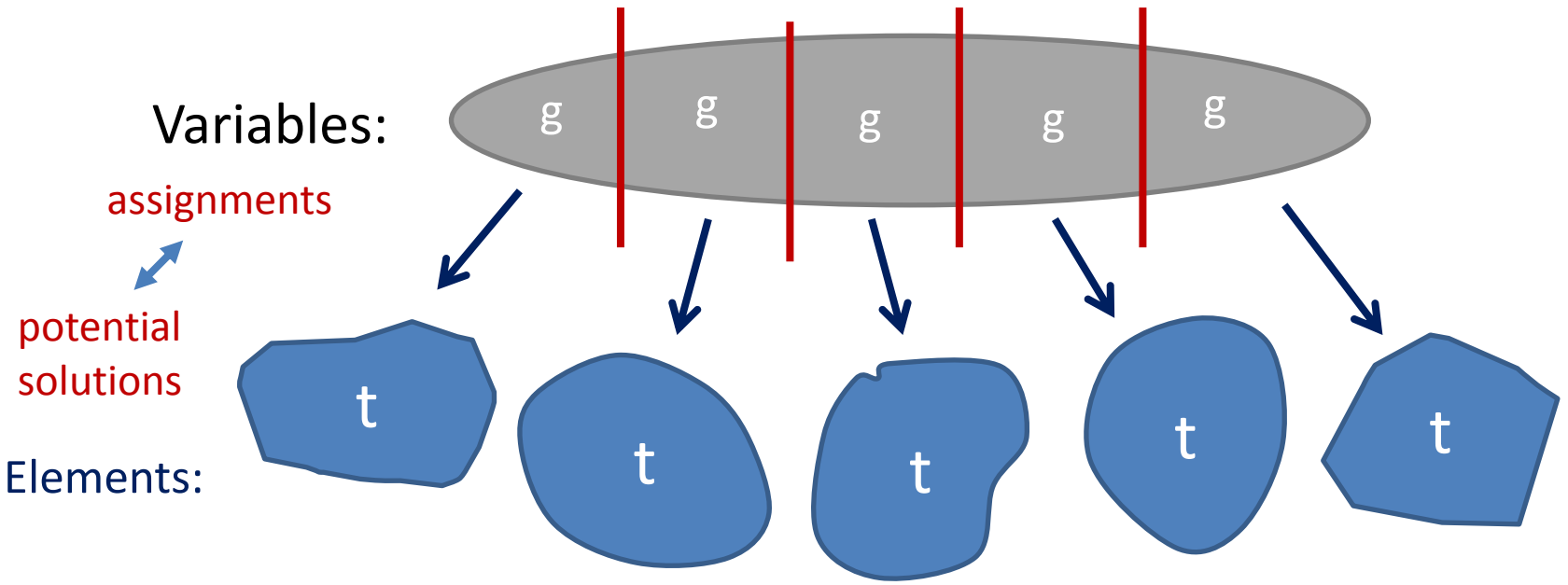
Sets of size $\lfloor t/2 \rfloor$

Adding a set of size $\lfloor t/2 \rfloor$ to the family F ensures that the «group complement» set is not picked.

All other sets of size $\lceil t/2 \rceil$ in the group may still be picked in solution.

Forbid sets of size $\lceil t/2 \rceil$ that do not correspond to assignments.

d-SAT ≤ Hitting Set



Want:

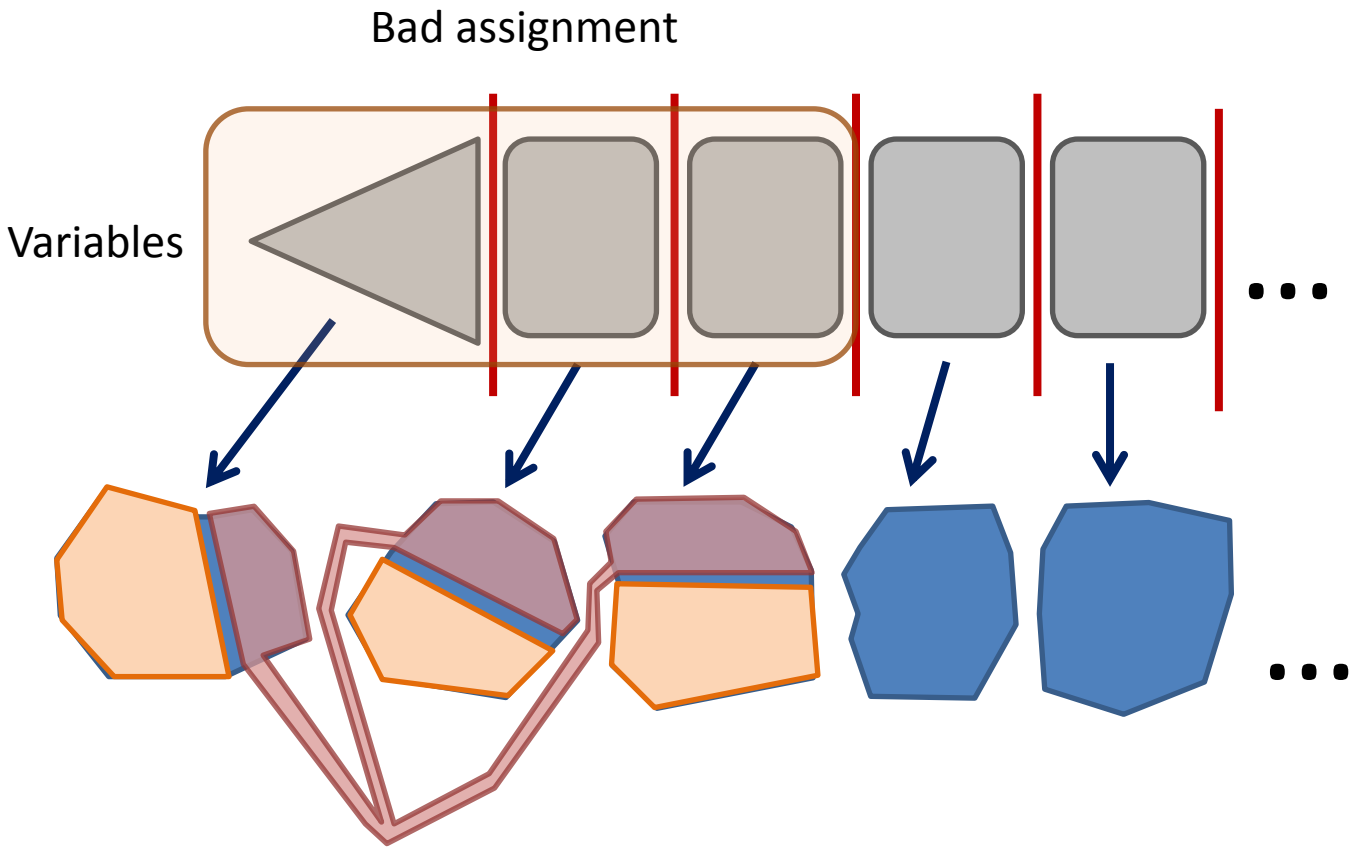
Solutions ↔ Satisfying assignments

Forbidding partial assignments

Pick any d groups of variables, and consider some assignment to these variables.

If this assignment falsifies ψ we want to forbid the corresponding set in the Hitting Set instance from being selected.

Forbidding partial assignments



Set added to F to forbid the bad assignment

Forbidding partial assignments

For each bad assignment to at most d groups, forbid it by adding a «bad assignment guard»

This adds $O(n^d 2^{gd}) = O(n^d)$ sets to F.

Satisfying Assignments ↔ Hitting Sets

A satisfying assignment has no bad sub-assignments → corresponds to a hitting set.

A hitting set corresponds to an assignment.

If this assignment falsified a clause C, the assignment would be bad for the $\leq$ d groups C lives in, and miss a bad assignment guard.

Hitting Set wrap up

Can reduce n variable d-SAT to $n(1+\epsilon)$ element Hitting Set.

So a c^n algorithm for Hitting Set yields a $(c+\epsilon)^n$ algorithm for d-SAT.

A 1.99^n algorithm for Hitting Set would violate SETH.

Conclusions

SETH can be used to give very tight running time bounds.

SETH recently has been used to give lower bounds for polynomial time solvable problems, and for running time of approximation algorithms.

Important Open Problems

Can we show a 2^n lower bound for Set Cover assuming SETH?

Can we show a 1.00001 lower bound for 3-SAT assuming SETH?

Exercises

Show that SETH implies ETH (Hint: use the sparsification lemma)

Exercise 4.16, 4.17, 4.18, 4.19.

Postdoc in Bergen?

Much better than Budapest. Really.

Thank You