Matroid Basics

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FPT School, Polland, August 18–22, 2014

Let us start with an example.

Kruskal's Greedy Algorithm for MWST

Let $G = (V, E)$ be a connected undirected graph and let $w : E \to \mathbb{R}^{\geq 0}$ be a weight function on the edges.
Kruskal's so-called greedy algorithm is as follows. The algorithm consists of selecting successively edges $e_1, e_2, \ldots, e_r$. If edges $e_1, e_2, \ldots, e_k$ has been selected, then an edge $e \in E$ is selected so that:

1. $e \notin \{e_1, \ldots, e_k\}$ and $\{e, e_1, \ldots, e_k\}$ is a forest.
2. $w(e)$ is as small as possible among all edges e satisfying (1).

We take $e_{k+1} := e$. If no e satisfying (1) exists then $\{e_1, \ldots, e_k\}$ is a spanning tree.

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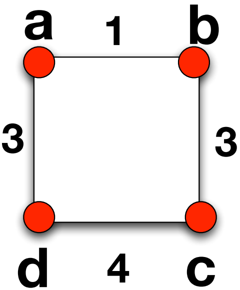
It is obviously not true that such a greedy approach would lead to an optimal solution for any combinatorial optimization problem.

Consider Maximum Weight Matching problem.



- Application of the greedy algorithm gives *cd* and *ab*.
- However, *cd* and *ab* do not form a matching of maximum weight.

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So a natural question is when does greedy works? Could one characterize for which set families greedy algorithm always outputs correct answer?

It turns out that the structures for which the greedy algorithm does lead to an optimal solution, are the *matroids*.

Matroids

Definition

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a *matroid* if it satisfies the following conditions:

(I1) $\varphi \in \mathcal{I}$ or $\mathcal{I} \neq \emptyset$.
(I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.
(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists\ e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the *hereditary property* and a pair $M = (E, \mathcal{I})$ satisfying (I1) and (I2) is called *hereditary family* or *set-family*.

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Rank and Basis

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An inclusion wise maximal set of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by $\text{rank}(M)$.

Matroids and Greedy Algorithms

Let $M = (E, \mathcal{I})$ be a set family and let $w : E \to \mathbb{R}^{\geq 0}$ be a weight function on the elements.

Objective: Find a set $Y \in \mathcal{I}$ that maximizes $w(Y) = \sum_{y \in Y} w(y)$.

The *greedy algorithm* consists of selecting successively $y_1, \ldots, y_r$ as follows. If $y_1, \ldots, y_k$ has been selected, then choose $y \in E$ so that:

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Matroids and Greedy Algorithms

THEOREM: A set family $M = (E, \mathcal{I})$ is a matroid

if and only if

the greedy algorithm leads to a set Y in $\mathcal{I}$ of maximum weight $w(Y)$, for each weight function $w : E \to \mathbb{R}^{\geq 0}$.

Examples Of Matroids

Uniform Matroid

A pair $M = (E, \mathcal{I})$ over an n-element ground set E, is called a *uniform matroid* if the family of independent sets is given by

$$\mathcal{I} = \Big\{ A \subseteq E \mid |A| \le k \Big\},$$

where k is some constant. This matroid is also denoted as $U_{n,k}$.
Eg: $E = \{1, 2, 3, 4, 5\}$ and $k = 2$ then

$$\mathcal{I} = \Big\{ \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \Big\}$$

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Partition Matroid

A *partition matroid* $M = (E, \mathcal{I})$ is defined by a ground set E being partitioned into (disjoint) sets $E_1, \ldots, E_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq E$ is independent if and only if $|X \cap E_i| \leq k_i$ for all $i \in \{1, \ldots, \ell\}$. That is,

$$\mathcal{I} = \Big\{ X \subseteq E \mid |X \cap E_i| \leq k_i, \ i \in \{1, \ldots, \ell\} \Big\}.$$

- If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, there must exist i such that $|Y \cap E_i| > |X \cap E_i|$ and this means that adding any element e in $E_i \cap (Y \setminus X)$ to X will maintain independence.
- M in general would not be a matroid if E_i were not disjoint. Eg: $E_1 = \{1, 2\}$ and $E_2 = \{2, 3\}$ and $k_1 = 1$ and $k_2 = 1$ then both $Y = \{1, 3\}$ and $X = \{2\}$ have at most one element of each E_i but one can't find an element of Y to add to X.

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Graphic Matroid

Given a graph G, a *graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \left\{ F \subseteq E(G) : \ F \text{ is a forest in the graph } G \right\}$$

Graphic Matroid: Why?

- Given two forests F_1 and F_2 such that $|F_2| > |F_1|$. Need to show that there is an edge $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\}$ is a forest.

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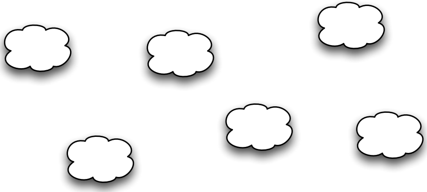

F_1

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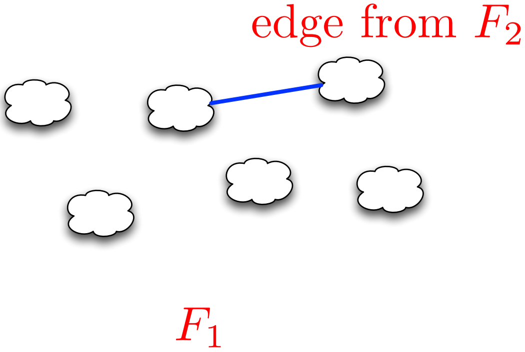

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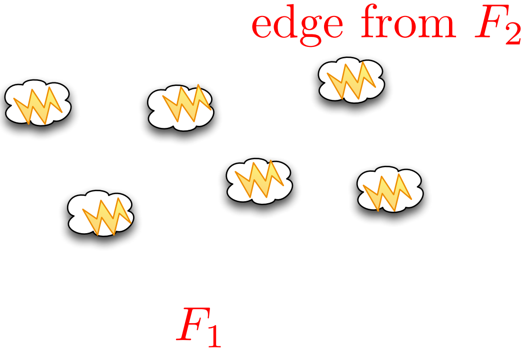

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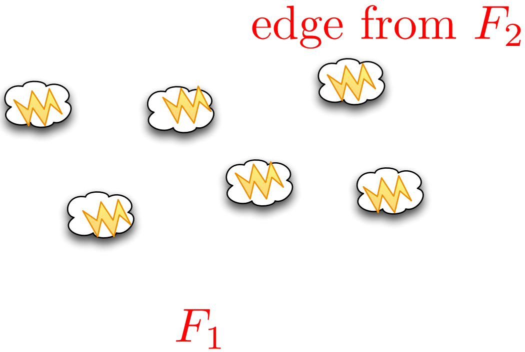
But this will imply that $|F_2| \leq |F_1|$ – a contradiction!



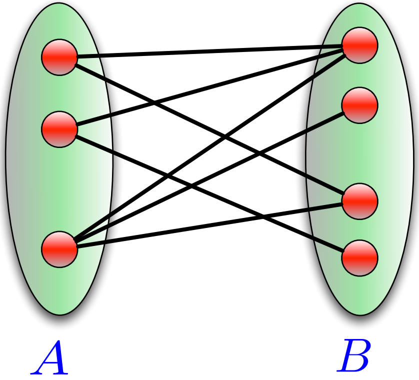
Co-Graphic Matroid

Given a graph G, a *co-graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \left\{ S \subseteq E(G) : \ G \setminus S \text{ is connected} \right\}$$

Transversal Matroid

Let G be a bipartite graph with the vertex set $V(G)$ being partitioned as A and B.



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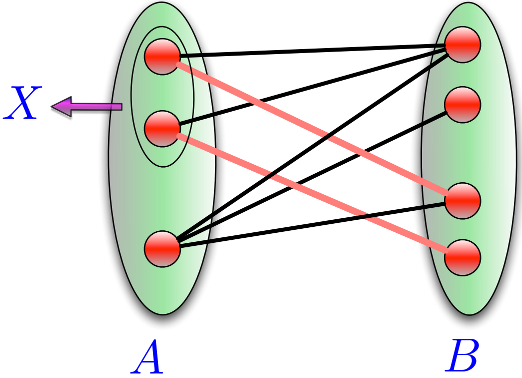
$$\mathcal{I} = \left\{ X \mid X \subseteq A, \text{ there is a matching that covers } X \right\}$$



Gammoids

Let $D = (V, A)$ be a directed graph, and let $S \subseteq V$ be a subset of vertices. A subset $X \subseteq V$ is *said to be linked to S* if there are $|X|$ vertex disjoint paths going from S to X.

The paths are disjoint, not only internally disjoint. Furthermore, zero-length paths are also allowed if $X \cap S = \emptyset$.

Given a digraph $D = (V, A)$ and subsets $S \subseteq V$ and $T \subseteq V$, a *gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = T$ and

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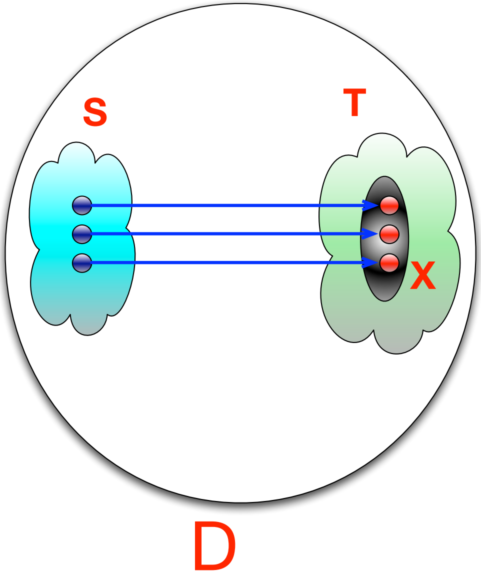
Gammoid: Example



Strict Gammoids

Given a digraph $D = (V, A)$ and subset $S \subseteq V$, a *strict gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = V$ and

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An Alternate Definition of Matroids

Basis Family

Let E be a finite set and $\mathcal{B}$ be a family of subsets of E that satisfies:

(B1) $\mathcal{B} \neq \emptyset$.
(B2) If $B_1, B_2 \in \mathcal{B}$ then $|B_1| = |B_2|$.
(B3) If $B_1, B_2 \in \mathcal{B}$ and there is an element $x \in (B_1 \setminus B_2)$ then there is an element $y \in (B_2 \setminus B_1)$ so that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

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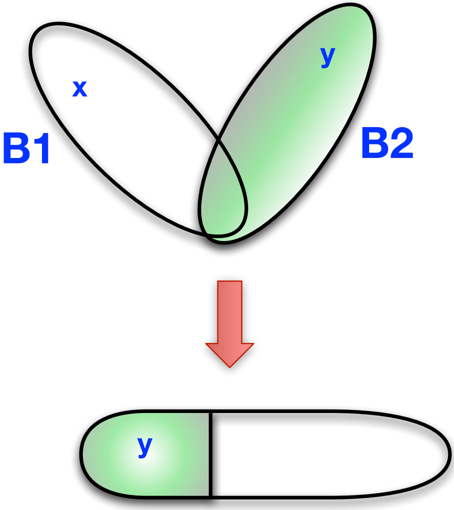

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Given $\mathcal{B}$, we define

$$\mathcal{I} = \mathcal{I}(\mathcal{B}) = \left\{ I \mid I \subseteq B \text{ for some } B \in \mathcal{B} \right\}$$

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Family of Bases

Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{B}$ be the family of bases of M – a family of maximal independent sets.
Then $\mathcal{B}$ satisfies (B1), (B2) and (B3). That is,

(B1) $\mathcal{B} \neq \emptyset$.
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(B3) If $B_1, B_2 \in \mathcal{B}$ and there is an element $x \in (B_1 \setminus B_2)$ then there is an element $y \in (B_2 \setminus B_1)$ so that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

Proof for B3

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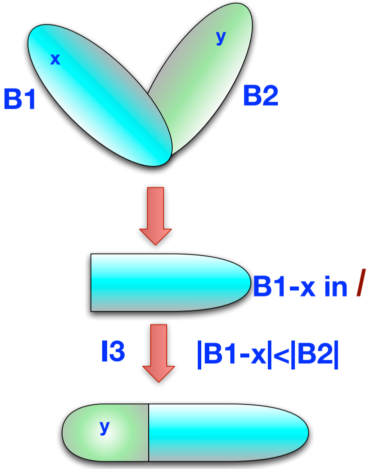

An Alternate Axiom System

THEOREM: Let E be a finite set and $\mathcal{B}$ be the family of subsets of E that satisfies (B1), (B2) and (B3) then $M = (E, \mathcal{I})$ is a matroid and $\mathcal{B}$ is the family of bases of this matroid. Recall, that

$$\mathcal{I} = \mathcal{I}(\mathcal{B}) = \left\{ I \mid I \subseteq B \text{ for some } B \in \mathcal{B} \right\}.$$

New Matroids from Old

Direct Sum

Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\cdots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$.

The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if for all $i \leq t$, $X \cap E_i \in \mathcal{I}_i$.

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$$\mathcal{I} = \left\{ X \mid X \subseteq E,\ (X \cap E_i) \in \mathcal{I}_i,\ i \in \{1, \ldots, t\} \right\}$$

Truncation

The *t-truncation* of a matroid $M = (E, \mathcal{I})$ is a matroid $M' = (E, \mathcal{I}')$ such that $S \subseteq E$ is independent in M' if and only if $|S| \leq t$ and S is independent in M (that is $S \in \mathcal{I}$).

Dual

Let $M = (E, \mathcal{I})$ be a matroid, $\mathcal{B}$ be the family of its bases and

$$\mathcal{B}^* = \Big\{E \setminus B \mid B \in \mathcal{B}\Big\}.$$

The *dual of* a matroid M is $M^* = (E, \mathcal{I}^*)$, where

$$\mathcal{I}^* = \Big\{X \mid X \subseteq B,\ B \in \mathcal{B}^*\}.$$

That is, $\mathcal{B}^*$ is a family of bases of M^*.

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Matroid Representation

Remark

- Need compact representation to for the family of independent sets.
- Also to be able to test easily whether a set belongs to the family of independent sets.

Linear Matroid

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. Given A we define the matroid $M = (E, \mathcal{I})$ as follows.
A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are *linearly independent* over $\mathbb{F}$.

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \quad * \text{ are elements of } F$$

The matroids that can be defined by such a construction are called *linear matroids*.

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Linear Matroids and Representable Matroids

If a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is *representable* over $\mathbb{F}$.

Linear Matroids and Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.

Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

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Linear Matroid

Let $M = (E, \mathcal{I})$ be linear matroid and Let $E = \{e_1, \ldots, e_m\}$ and d=rank(M).

We can always assume (using Gaussian Elimination) that the matrix has following form:

$$\left[\begin{array}{c|c} I_{d\times d} & D \end{array}\right]_{d\times m}$$

Here $I_{d\times d}$ is a $d \times d$ identity matrix.

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Direct Sum of Matroid

Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\cdots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$. The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if for all $i \leq t$, $X \cap E_i \in \mathcal{I}_i$.
Let A_i be the representation matrix of $M_i = (E_i, \mathcal{I}_i)$ over the same field $\mathbb{F}$. Then,

$$A_M = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_t \end{pmatrix}$$

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Deletion of a Matroid

Let $M = (E, \mathcal{I})$ be a matroid, and let X be a subset of E. Deleting X from M gives a matroid $M \setminus X = (E \setminus X, \mathcal{I}')$ such that $S \subseteq E \setminus X$ is independent in $M \setminus X$ if and only if $S \in \mathcal{I}$.

$$\mathcal{I}' = \left\{ S \mid S \subseteq E \setminus X, S \in \mathcal{I} \right\}$$

Given a representation of A_M of M, a representation of $M \setminus X$ can be obtained by deleting the columns corresponding to X.

$$A_M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{array} \right]_{d \times m} \end{array}$$

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Deletion of a Matroid

Let $X = \{e_2, e_3\}$.

$$A_M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{array}\right]_{d \times m} \end{array}$$

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Dual of a Matroid

Let $M = (E, \mathcal{I})$ be a matroid, $\mathcal{B}$ be the family of its bases and

$$\mathcal{B}^* = \left\{ E \setminus B \mid B \in \mathcal{B} \right\}.$$

The *dual of* a matroid M is $M^* = (E, \mathcal{I}^*)$, where

$$\mathcal{I}^* = \left\{ X \mid X \subseteq B,\ B \in \mathcal{B}^* \right\}.$$

That is, $\mathcal{B}^*$ is a family of bases of M^*.
Let $A = [I_{d \times d} \mid D]$ represent the matroid M then the matrix $A^* = [-D^T \mid I_{m-r \times m-r}]$ represents the dual matroid M^*.

Dual of a Matroid: A concrete example

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\{a, b, c, d\}$ is a basis of M then $\{e, f, g\}$ is a basis of M^*.

$$A^* = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{bmatrix} & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{bmatrix} \end{array}$$

To find coordinates for columns a, b, c, d, we will choose entries that make *every row of A orthogonal to every row of A^*.*

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Here, D is colored in green.

Uniform Matroid

Every uniform matroid is linear and can be represented over a finite field by a $k \times n$ matrix A_M where the $A_M[i,j] = j^{i-1}$.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 3 & \cdots & n \\
1 & 2^2 & 3^2 & \cdots & n^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1}
\end{bmatrix}_{k \times n}
\end{array}
$$

Observe that for A_M to be representable over a finite field $\mathbb{F}$, we need that the determinant of any $k \times k$ submatrix of A_M must not vanish over $\mathbb{F}$.

The determinant of any $k \times k$ submatrix of A_M is upper bounded by $k! \times n^{k-1}$ (this follows from the Laplace expansion of determinants). Thus, choosing a field $\mathbb{F}$ of size larger than $k! \times n^{k-1}$ suffices.

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Uniform Matroid: Size of the representation

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So the size of the representation: $O((k \log n) \times nk)$ bits.

Graphic Matroid

The graphic matroid is representable over any field of size at least 2.

Consider the matrix A_M with a row for each vertex $i \in V(G)$ and a column for each edge $e = ij \in E(G)$. In the column corresponding to $e = ij$, all entries are 0, except for a 1 in i or j.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1 \end{array}\right]_{n \times |E(G)|} \end{array}$$

This is a representation over $\mathbb{F}_2$.

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Graphic Matroid?

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- Let X be a minimal set of columns that are linearly dependent then the corresponding edges form ?

Transversal Matroid

For the bipartite graph with partition A and B, form an incidence matrix A_M as follows. Label the rows by vertices of B and the columns by the vertices of A_M and define:

$$a_{ij} = \begin{cases} z_{ij} & \text{if there is an edge between } a_i \text{ and } b_j, \\ 0 & \text{otherwise.} \end{cases}$$

where z_{ij} are in-determinants. Think of them as independent variables.

$$T = \begin{array}{c} \\ b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & \cdots & a_j & \cdots & a_\ell \end{array} \\ \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1j} & \cdots & z_{1\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{i1} & z_{i2} & \cdots & z_{ij} & \cdots & z_{i\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nj} & \cdots & z_{n\ell} \end{bmatrix} \end{array}$$

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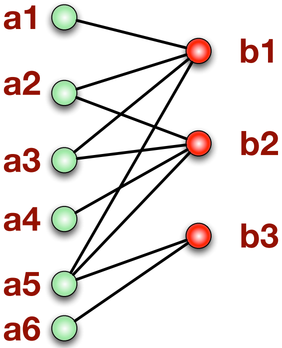
Example of the Construction



$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{bmatrix} z_{11} & z_{12} & z_{13} & 0 & z_{15} & 0 \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} & 0 \\ 0 & 0 & 0 & 0 & z_{35} & z_{36} \end{bmatrix} \end{array}$$

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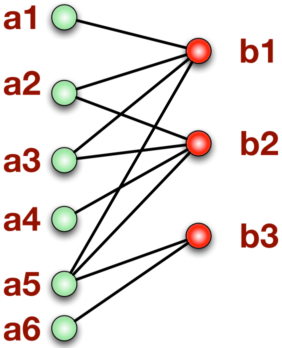

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\end{array}
$$

Permutation expansion of Determinants

THEOREM: Let

$$A = (a_{ij})_{n \times n}$$

be a $n \times n$ matrix with entries in $\mathbb{F}$. Then

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)}.$$

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Forward direction:

- Suppose some subset $X = \{a_1, \ldots, a_q\}$ is independent.
- Then there is a matching that saturates X. Let $Y = \{b_1, b_2, \ldots, b_q\}$ be the endpoints of this matching and $a_i b_i$ are the matching edges.
- Consider $T[Y, X]$ – a submatrix with rows in Y and columns in X. Consider the determinant of $T[Y, X]$ then we have a term

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which can not be cancelled by any other term! So these columns are linearly independent.

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- Then there is a submatrix of $T[\star, X]$ that has non-zero determinant – say $T[Y, X]$.
- Consider the determinant of $T[Y, X]$

$$0 \neq \det(T[Y,X]) = \sum_{\pi \in S(Y)} \operatorname{sgn}(\pi) \prod_{i=1}^{q} z_{i\pi(i)}.$$

- This implies that we have a term

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- For this direction we do not use z_{ij}, the very fact that X forms independent set of column is enough to argue that there is a matching that saturates X.

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Reverse direction:

- This implies that we have a term

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  and this gives us that there is a matching that saturates X in and hence X is independent.
- For this direction we do not use z_{ij}, the very fact that X forms independent set of column is enough to argue that there is a matching that saturates X.

Removing z_{ij}

To remove the z_{ij} we do the following.

Uniformly at random assign z_{ij} from values in finite field $\mathbb{F}$ of size P.

What should be the upper bound on P? What is the probability that the randomly obtained T is a representation matrix for the transversal matroid.

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Using Zippel-Schwartz Lemma

THEOREM: Let $p(x_1, x_2, \ldots, x_n)$ be a non-zero polynomial of degree d over some field $\mathbb{F}$ and let S be an N element subset of $\mathbb{F}$. If each x_i is independently assigned a value from S with uniform probability, then $p(x_1, x_2, \ldots, x_n) = 0$ with probability at most $\frac{d}{N}$.

- We think $\det(T[Y, X])$ as polynomial in z_{ij}'s of degree at most $n = |A|$.
- Probability that $\det(T[Y, X]) = 0$ is less than $\frac{n}{P}$. There are at most 2^n independent sets in A and thus by union bound probability that not all of them are independent in the matroid represented by T is at most $\frac{2^n n}{P}$.

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- Thus probability that T is the representation is at least $1 - \frac{2^n n}{P}$. Take P to be some field with at least $2^n n 2^n$ elements :-).
- size of this representation with be like $n^{O(1)}$ bits!

Truncation of Matroid

Given a representable matroid:

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{array} \right]_{\ell \times m} \end{array}$$

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find t-truncation of this matroid.

- Idea is to take a random matrix say N (that is a matrix with entries chosen randomly from some sufficiently large field) of dimension $t \times \ell$.
- Compute NM and get M_t – a randomized representation for t-truncation.

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This is an important tool in parameterized algorithms — as this allows us to reduce the rank of the input matroid.

Correctness

- Use Zippel-Schwartz Lemma again!

Final Slide

Thank You!

Any Questions?