Lower bounds for polynomial kernelization

Part 1

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Outline

- **Goal**: how to prove that some problems do **not** admit polynomial kernelization algorithms?
- **Part 1:**
  - Introduction of the (cross)-composition framework.
  - Basic examples.
- **Part 2**:
  - PPT reductions.
  - Case study of several cross-compositions.
  - Weak compositions.

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- *Unparameterized variant*: k is appended to x in unary.

Kernelization — recap

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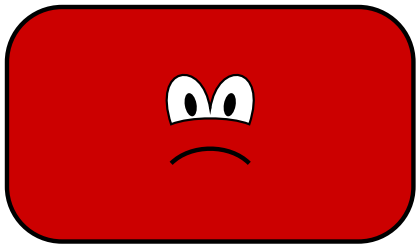

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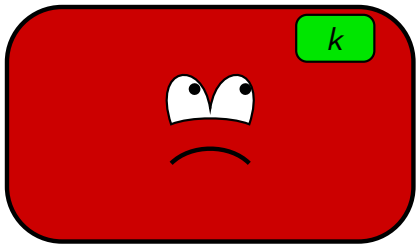

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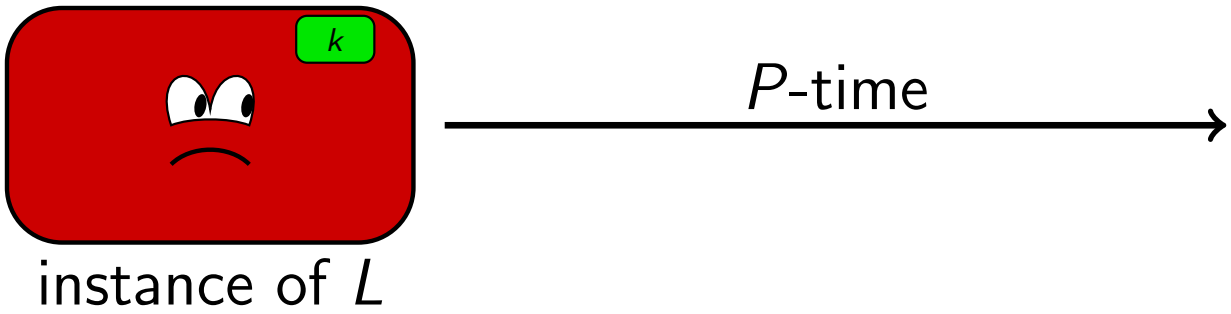

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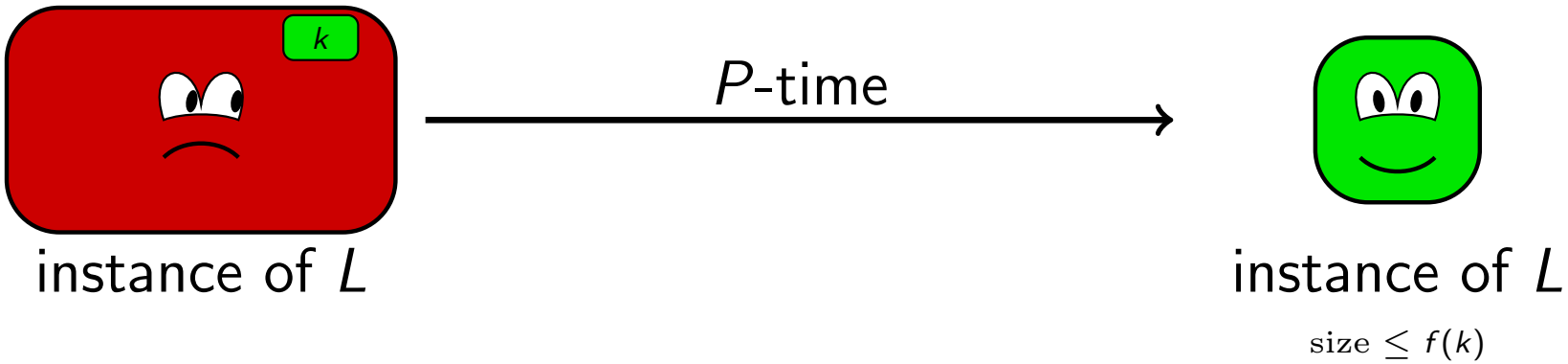

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- We are interested in **polynomial kernels**, where f is a polynomial.
- Before 2008, no tool to classify FPT problems wrt. whether they have polykernels or not.

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- Let H be a disjoint union of $G_1, G_2, \ldots, G_t$. Then the answer to (H, k) is YES if and only if the answer to any (G_i, k) is YES.
- Apply kernelization to (H, k) obtaining an instance with k^3 vertices, encodable in k^6 bits.

Motivating intuition

- **Intuition**: The final number of bits is much less than the number input instances. Most of the instances must have been **discarded completely**.

Kernelization and Compression

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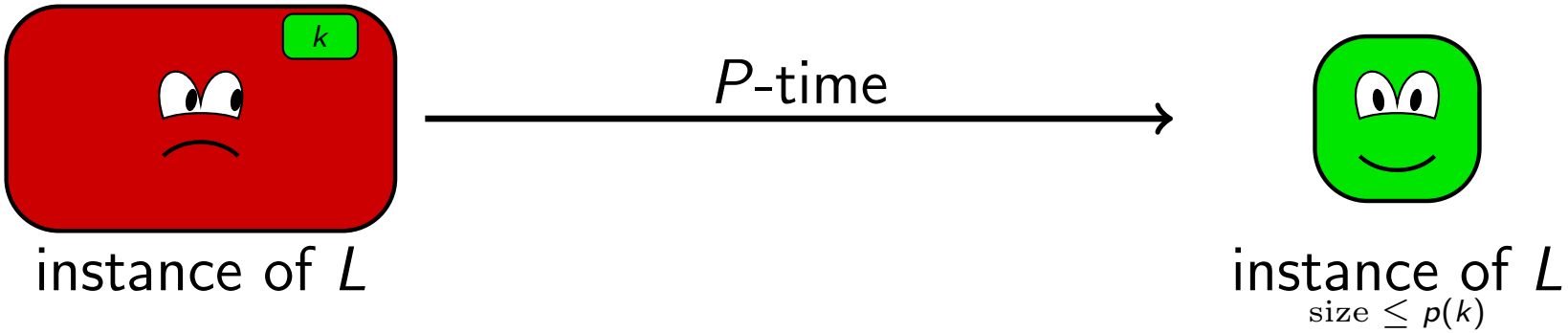
KERNELIZATION



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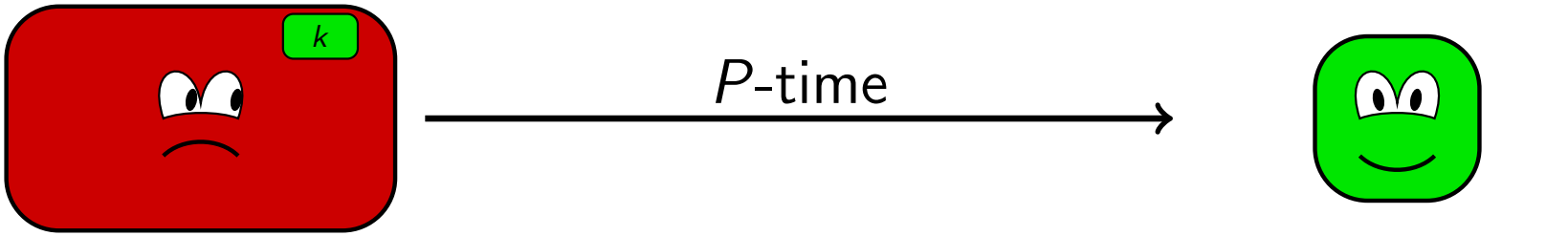

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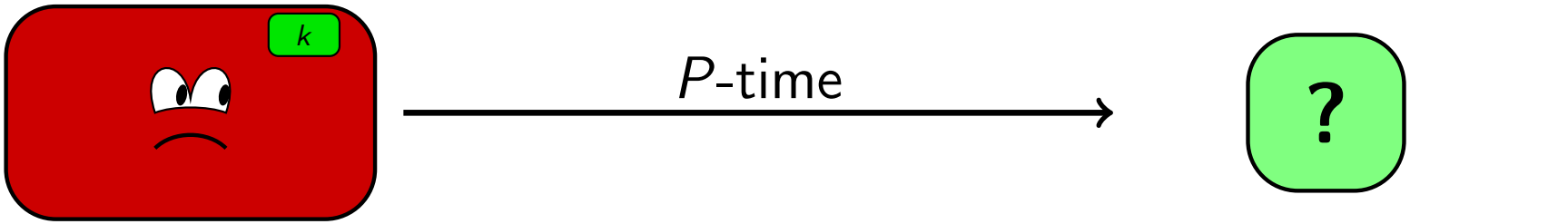

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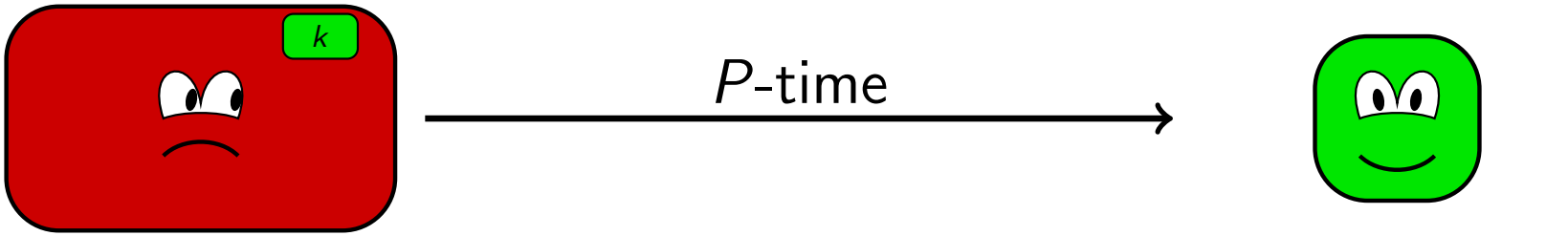

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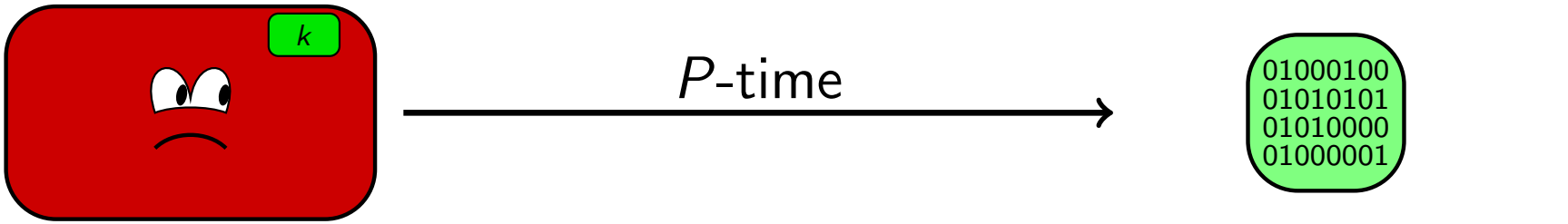

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- A polynomial compression can be turned into a polynomial kernelization provided that there is a **P**-reduction from R to L.
  - For instance, when $R \in \mathbf{NP}$ and L is **NP**-hard.
- **Note**: There are examples when a poly-compression is known but a poly-kernel is not known, because it is unclear whether R is in **NP**.

OR-distillation

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OR-distillation of L into R

Input: Strings $x_1, x_2, \ldots, x_t$, each of length at most k.
Time: $\mathrm{poly}(t + \sum_{i=1}^{t} |x_i|)$.
Output: One string y such that
(a) $|y| = \mathrm{poly}(k)$, and
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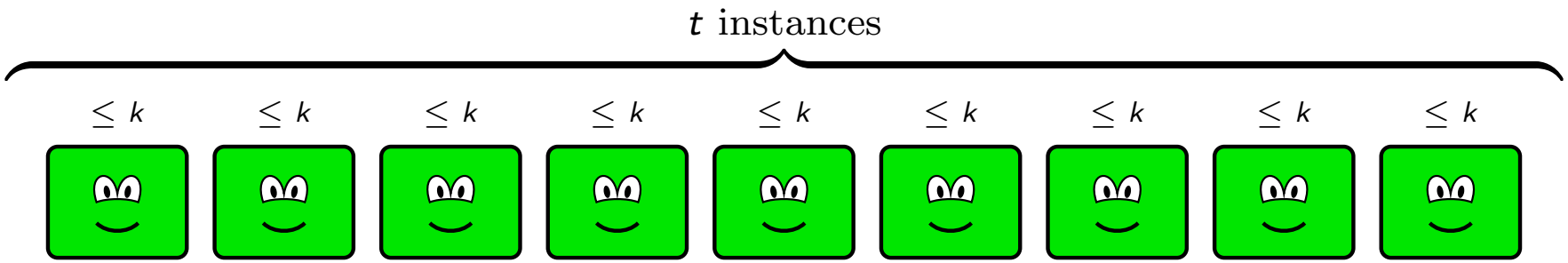

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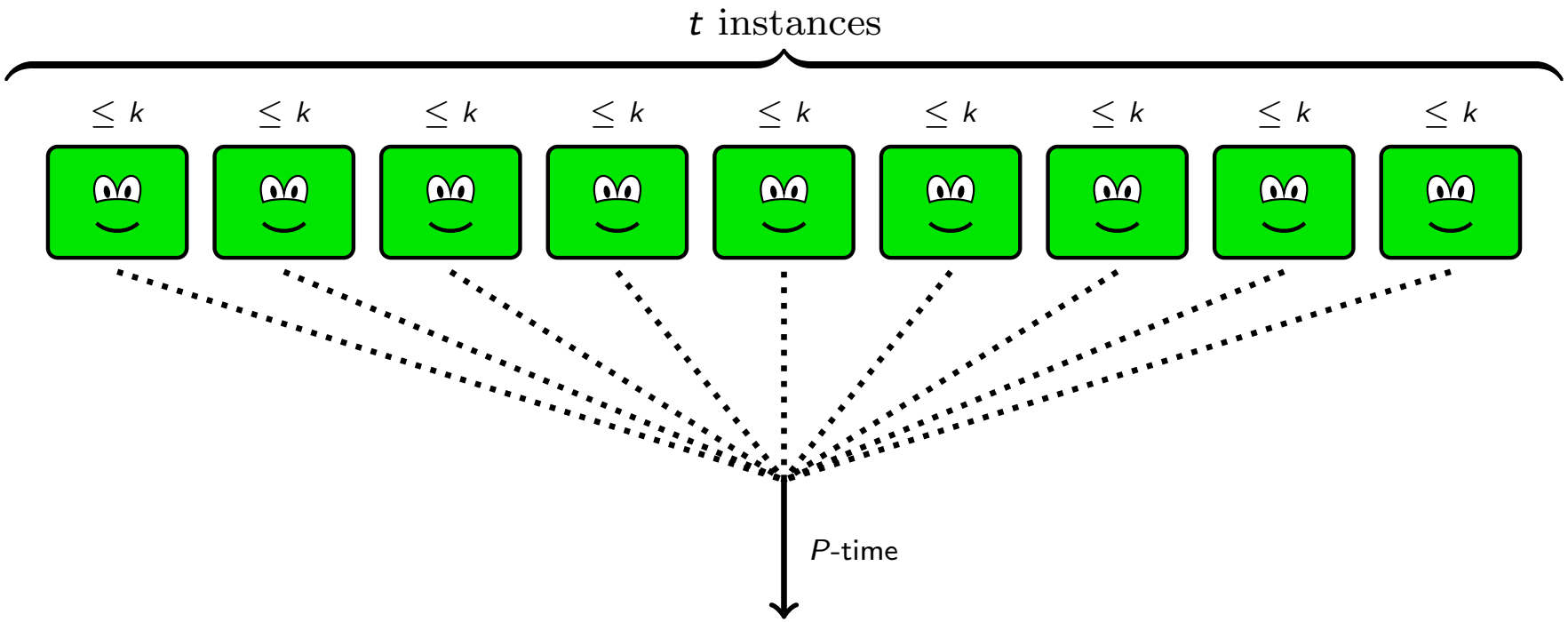

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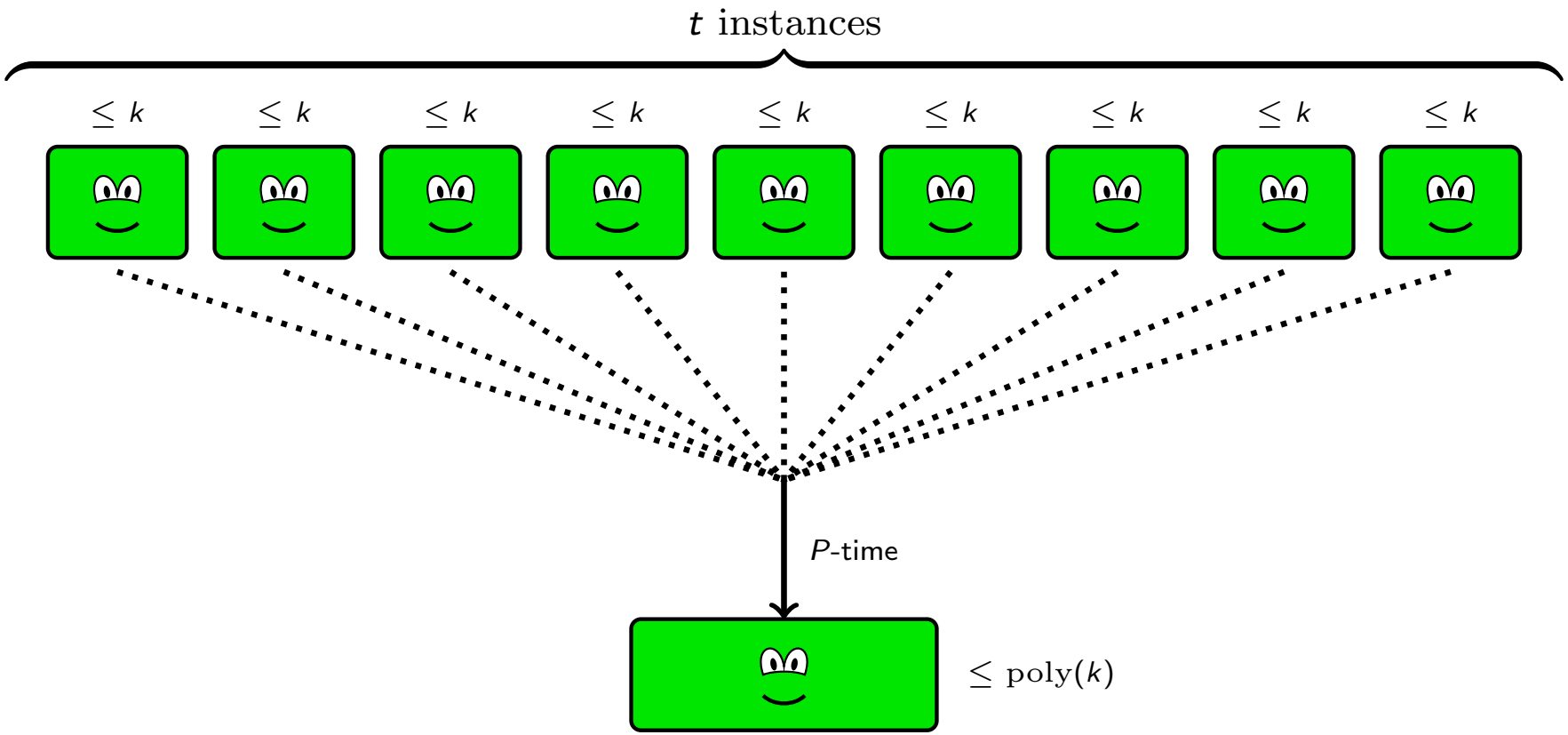

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- OR-L: language of strings $x_1\#x_2\#\ldots\#x_t$ such that $x_i \in L$ for at least one i.
- OR-distillation of L into R is a polynomial compression of OR-L into R, where OR-L is parameterized by $\max |x_i|$.

Backbone theorem

OR-distillation theorem Fortnow, Santhanam; STOC 2008, JCSS 2011

SAT does not admit an OR-distillation algorithm into any language R, unless $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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Corollary

No **NP**-hard problem admits an OR-distillation algorithm into any language R, unless $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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- It is known that $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$ implies that $\mathrm{PH} = \Sigma_3^{\mathrm{P}}$.
- Not as bad as $\mathbf{P} = \mathbf{NP}$, but pretty severe.

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- **Look into the book.**

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OR-composition algorithm for L

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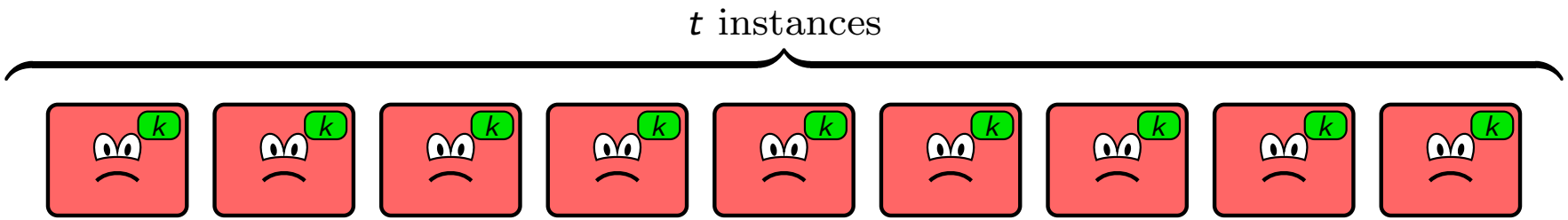

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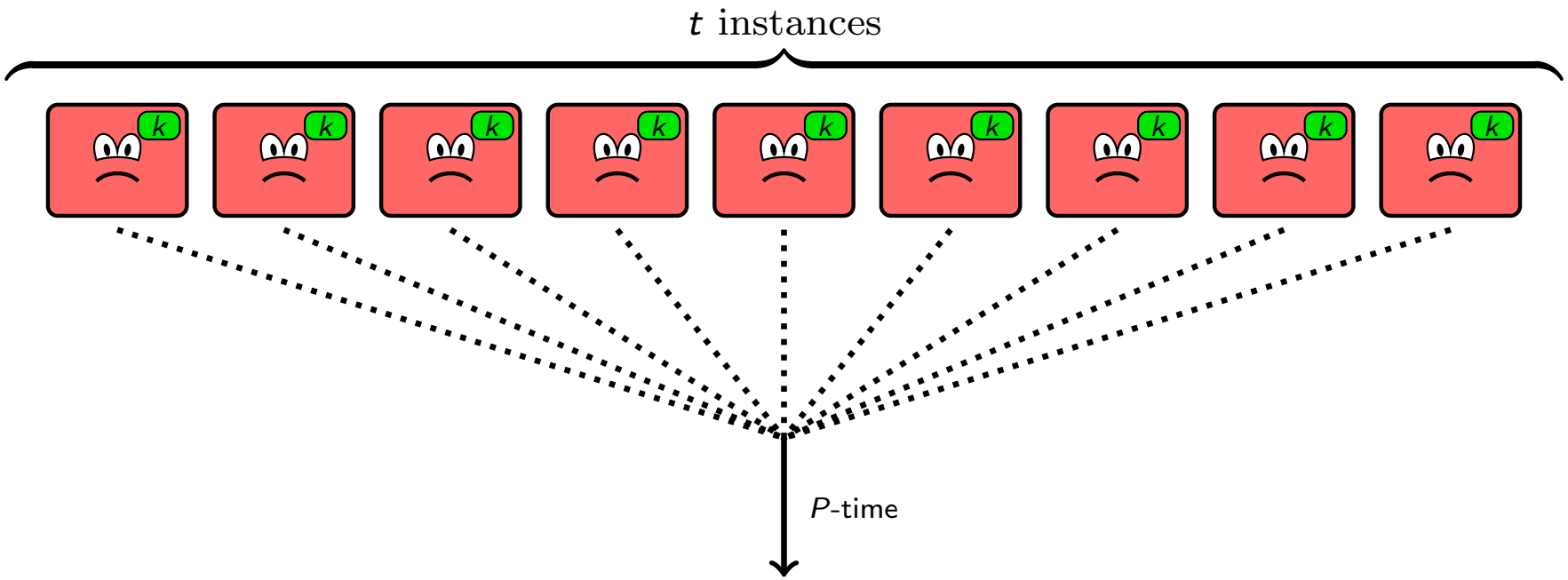

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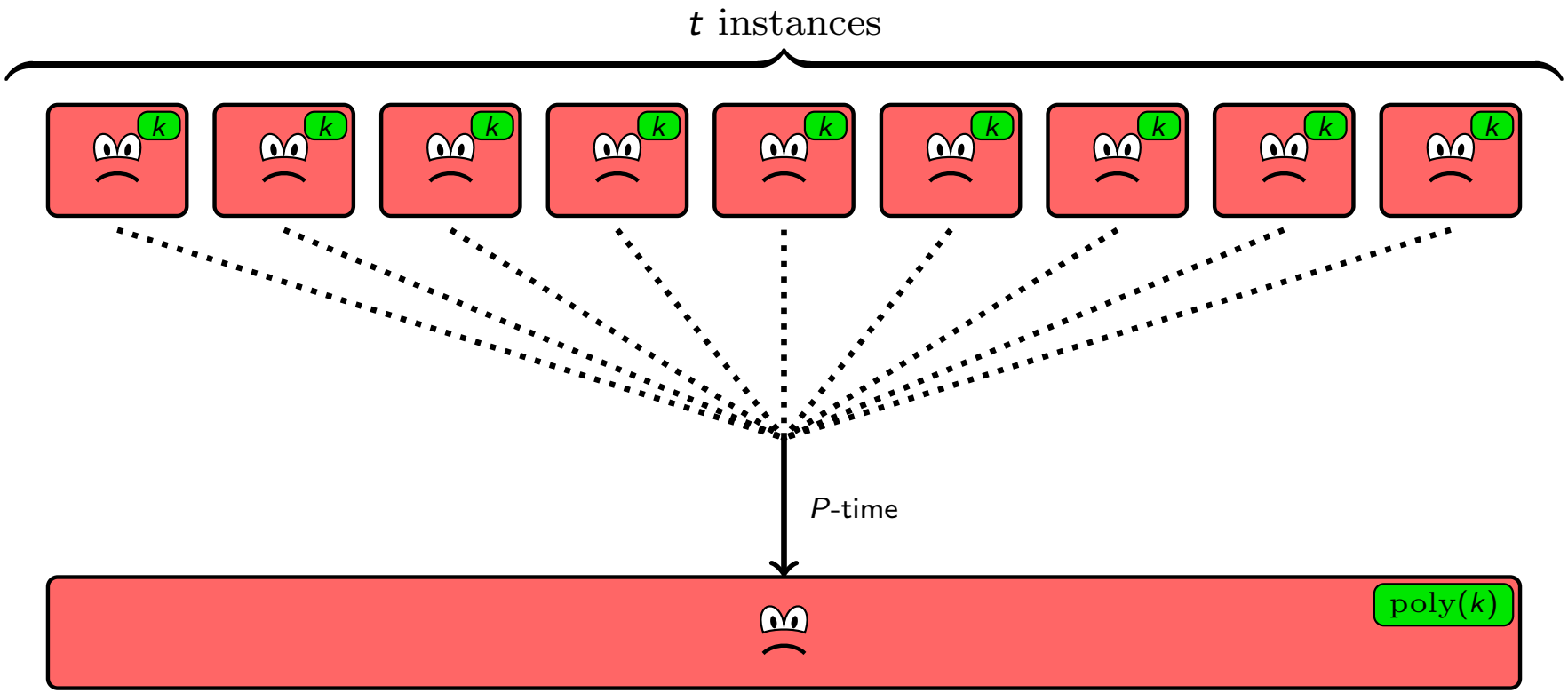

OR-composition theorem

OR-composition theorem Bodlaender et al.; ICALP 2008, JCSS 2009

If a parameterized problem L admits an OR-composition algorithm, and the unparameterized version of L is **NP**-hard, then L does not admit a polynomial kernel unless $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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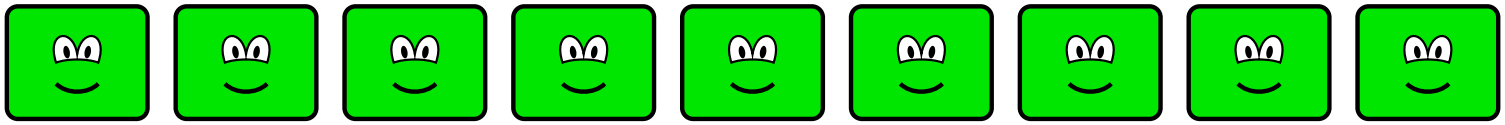
OR-SAT



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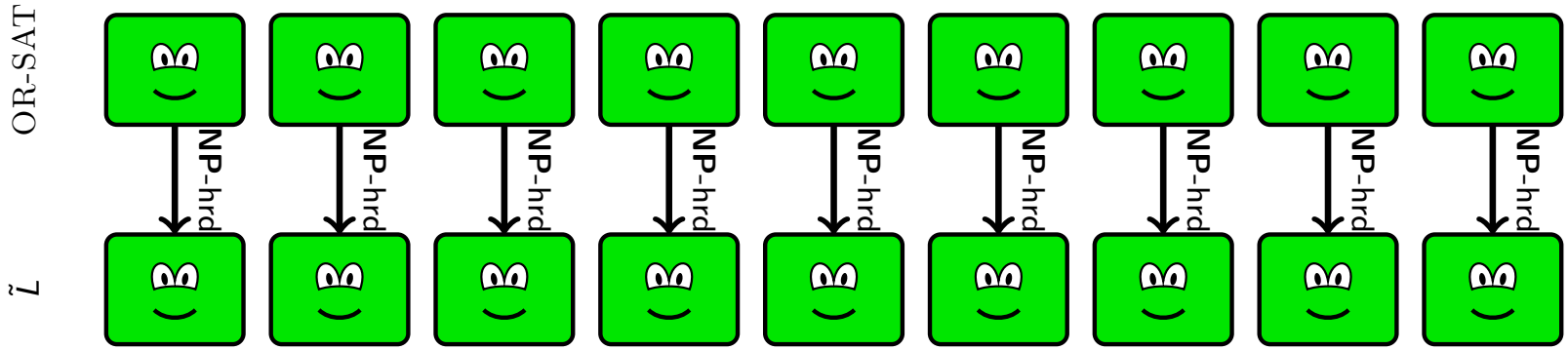
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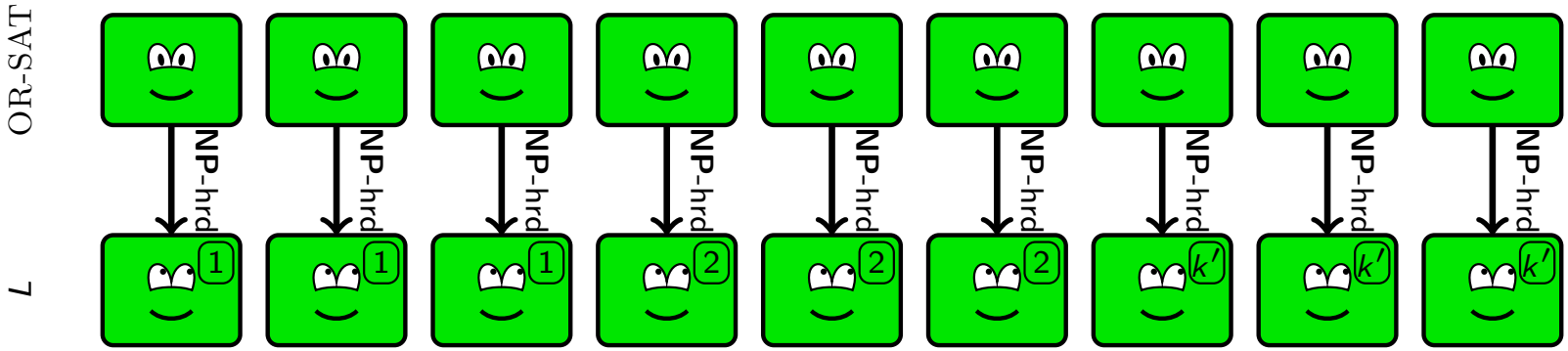

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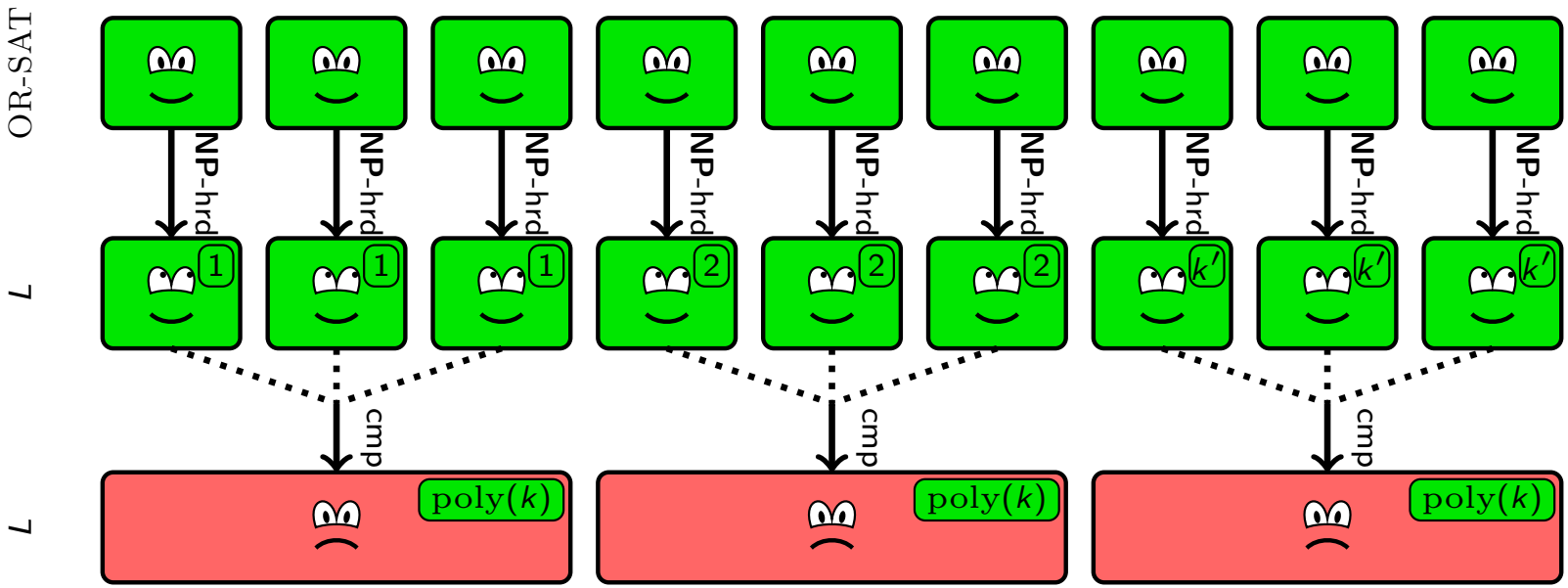
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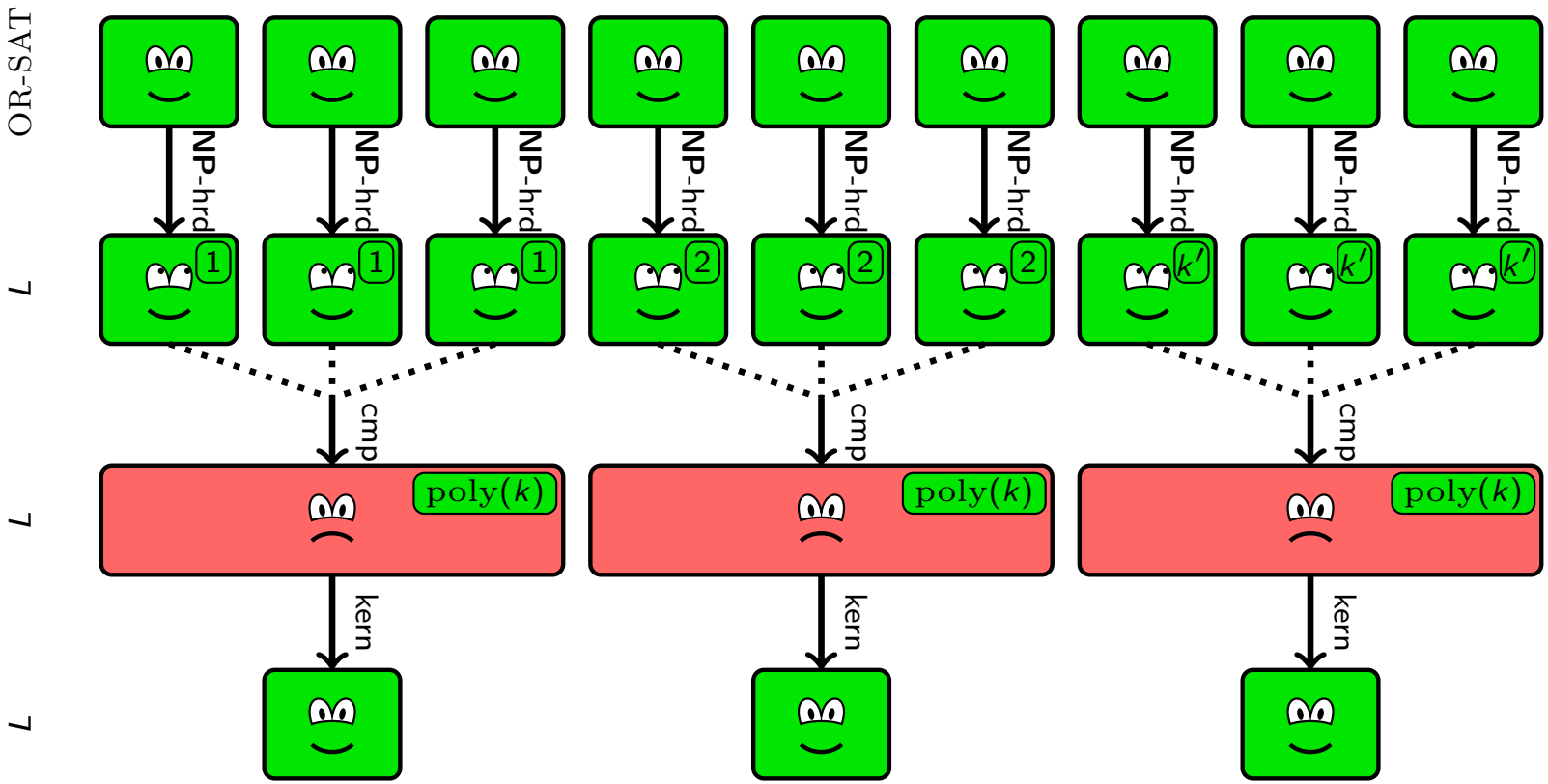
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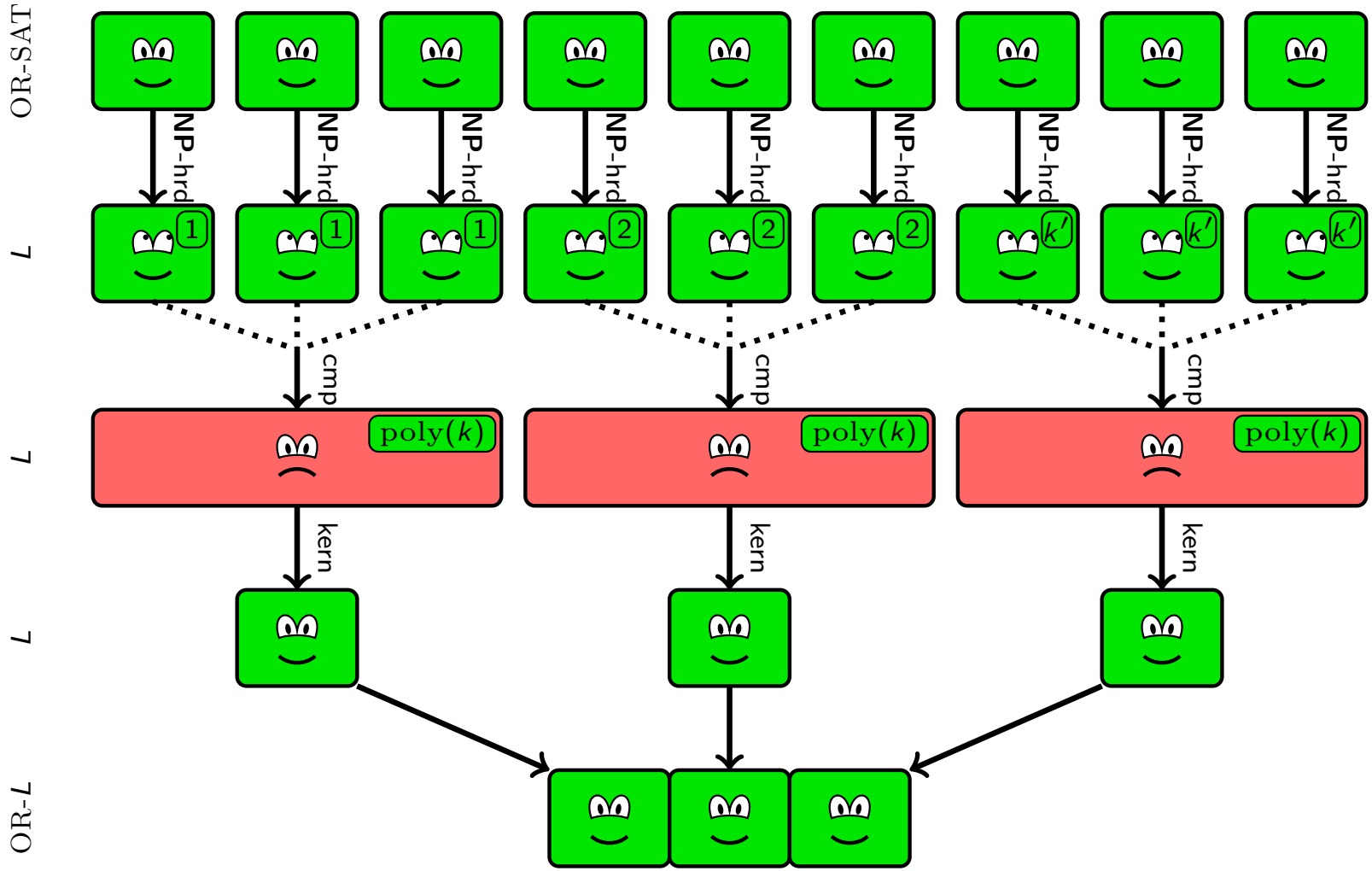

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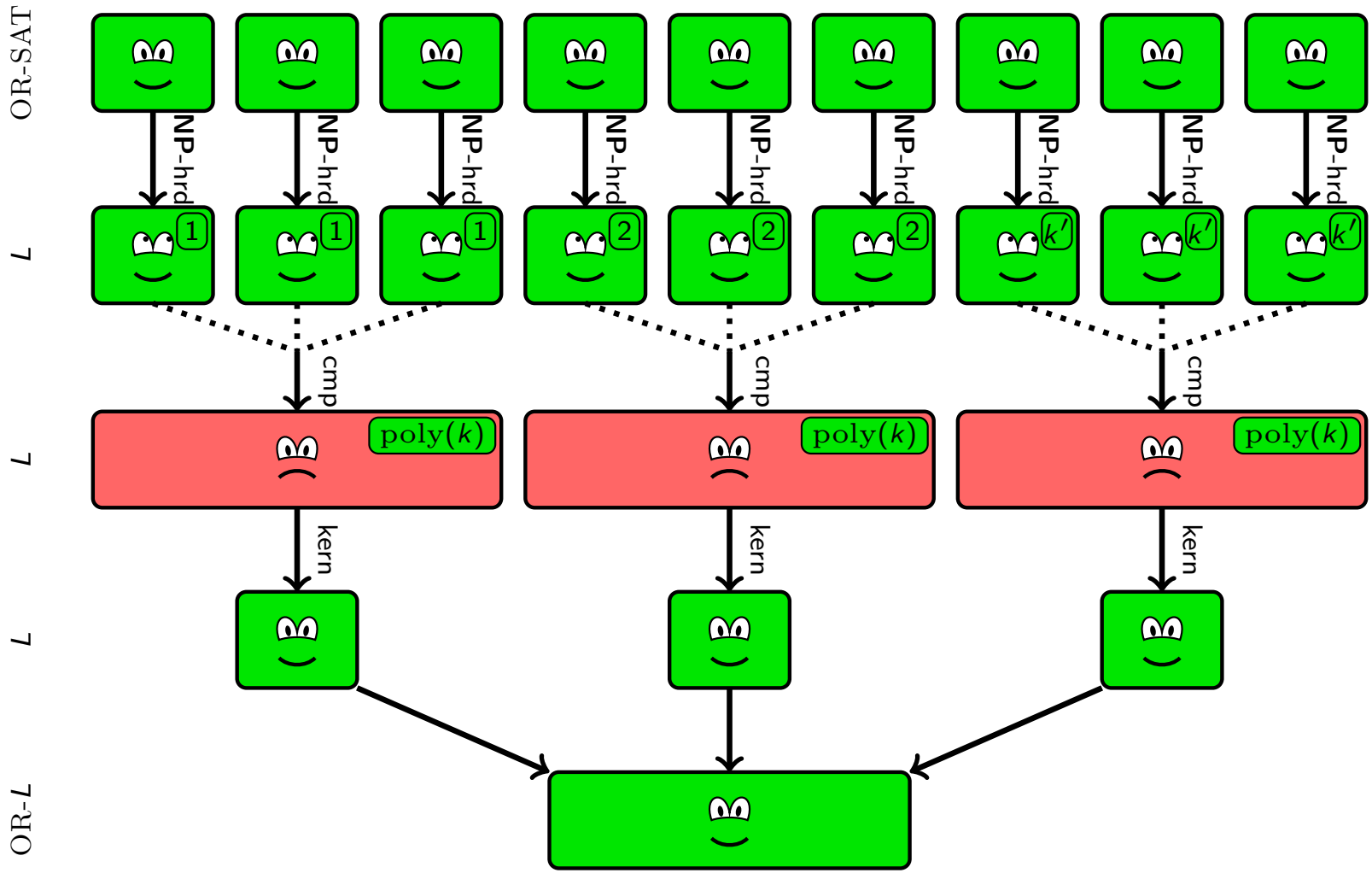
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- Same for k-CYCLE; this opens a bag of results.
- Today, investigating the existence of a polynomial kernel is an immediate second goal after showing that a problem is FPT.

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- Observed also earlier via different arguments (Dom, Lokshtanov, and Saurabh; ICALP 2009).

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- STACS 2011: Bodlaender, Jansen, and Kratsch propose a new formalism, dubbed **cross-composition**, that gathers all these features.

Polynomial equivalence relation

Polynomial equivalence relation

An equivalence relation $\mathcal{R}$ on $\Sigma^\star$ is called a polynomial equivalence relation if the following two conditions hold:

- Checking whether two strings $x, y \in \Sigma^\star$ are $\mathcal{R}$-equivalent can be done in $\mathrm{poly}(|x| + |y|)$ time.
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  - partitioning with respect to the number of vertices of the graph;
  - or with respect to *(i)* the number of vertices, *(ii)* the number of edges, *(iii)* size of the maximum matching, *(iv)* budget.

Cross-composition

Cross-composition

An unparameterized problem Q *cross-composes* into a parameterized problem L, if there exists a polynomial equivalence relation $\mathcal{R}$ and an algorithm that, given $\mathcal{R}$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

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Cross-composition theorem

Bodlaender et al.; STACS 2011, SIDMA 2014

If some **NP**-hard problem Q cross-composes into L, then L does not admit a polynomial compression into any language R, unless $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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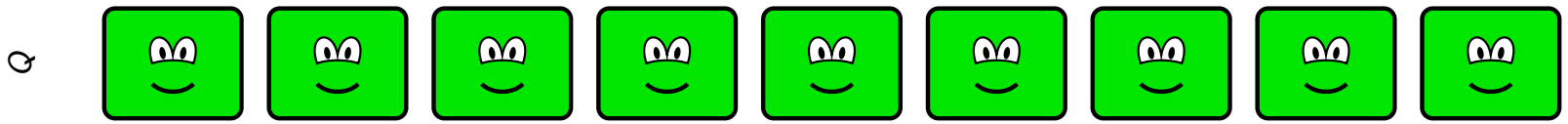
$k = \max |x_i|, \quad \log t = \mathcal{O}(k)$



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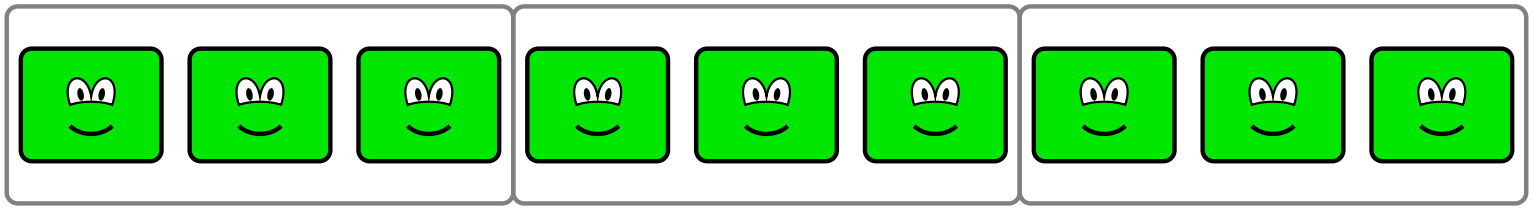

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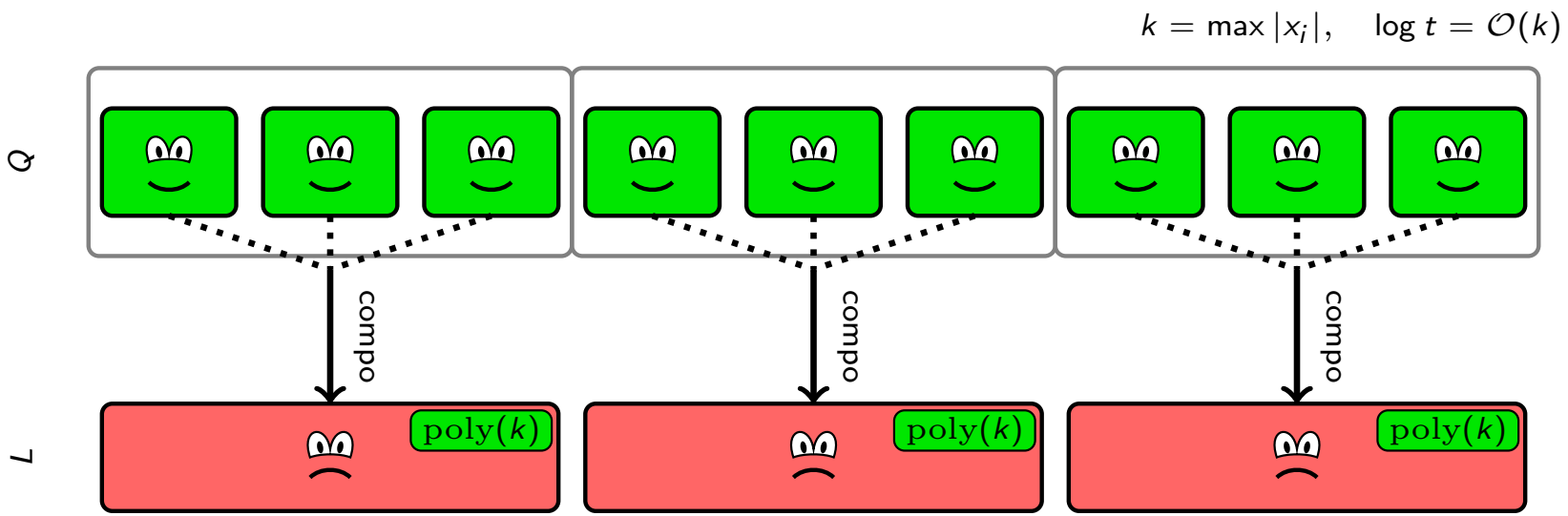
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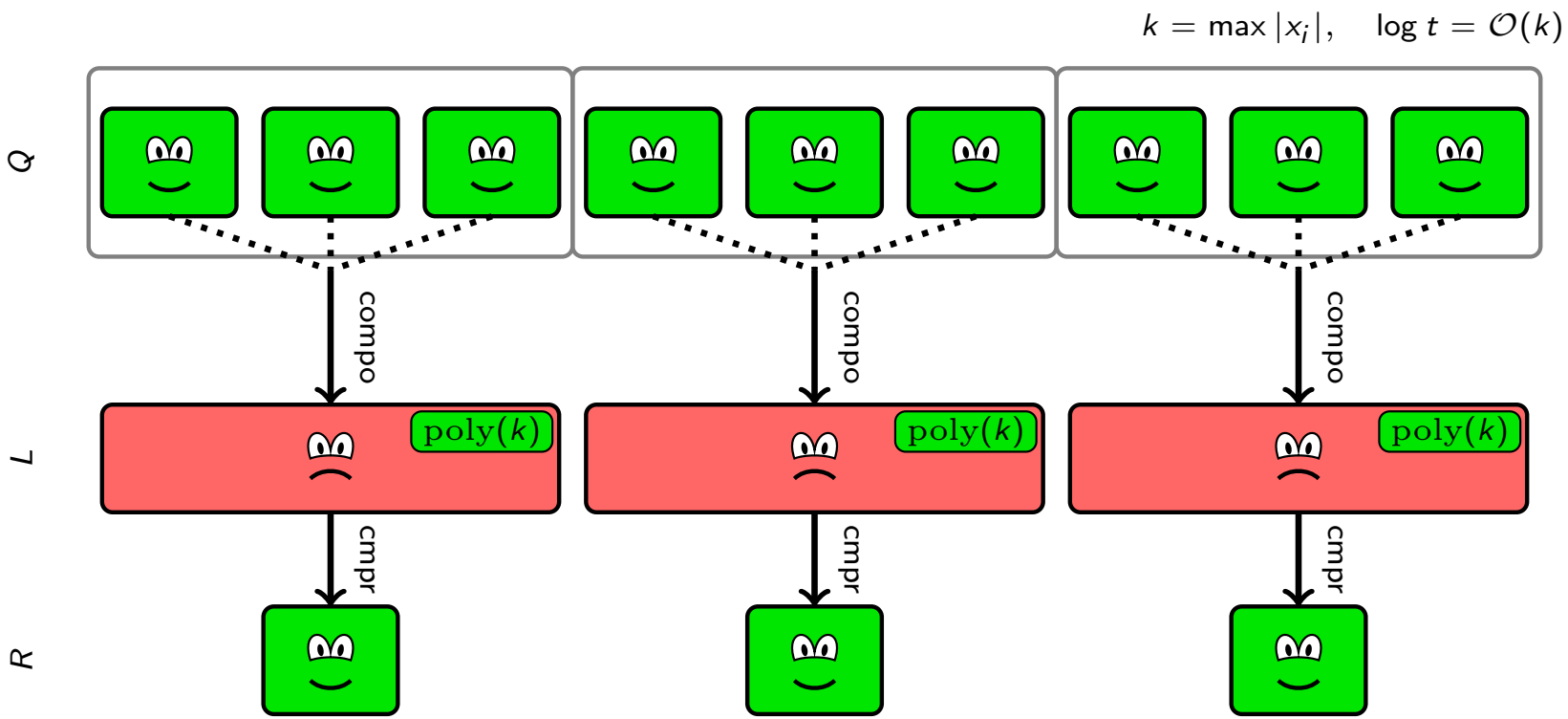
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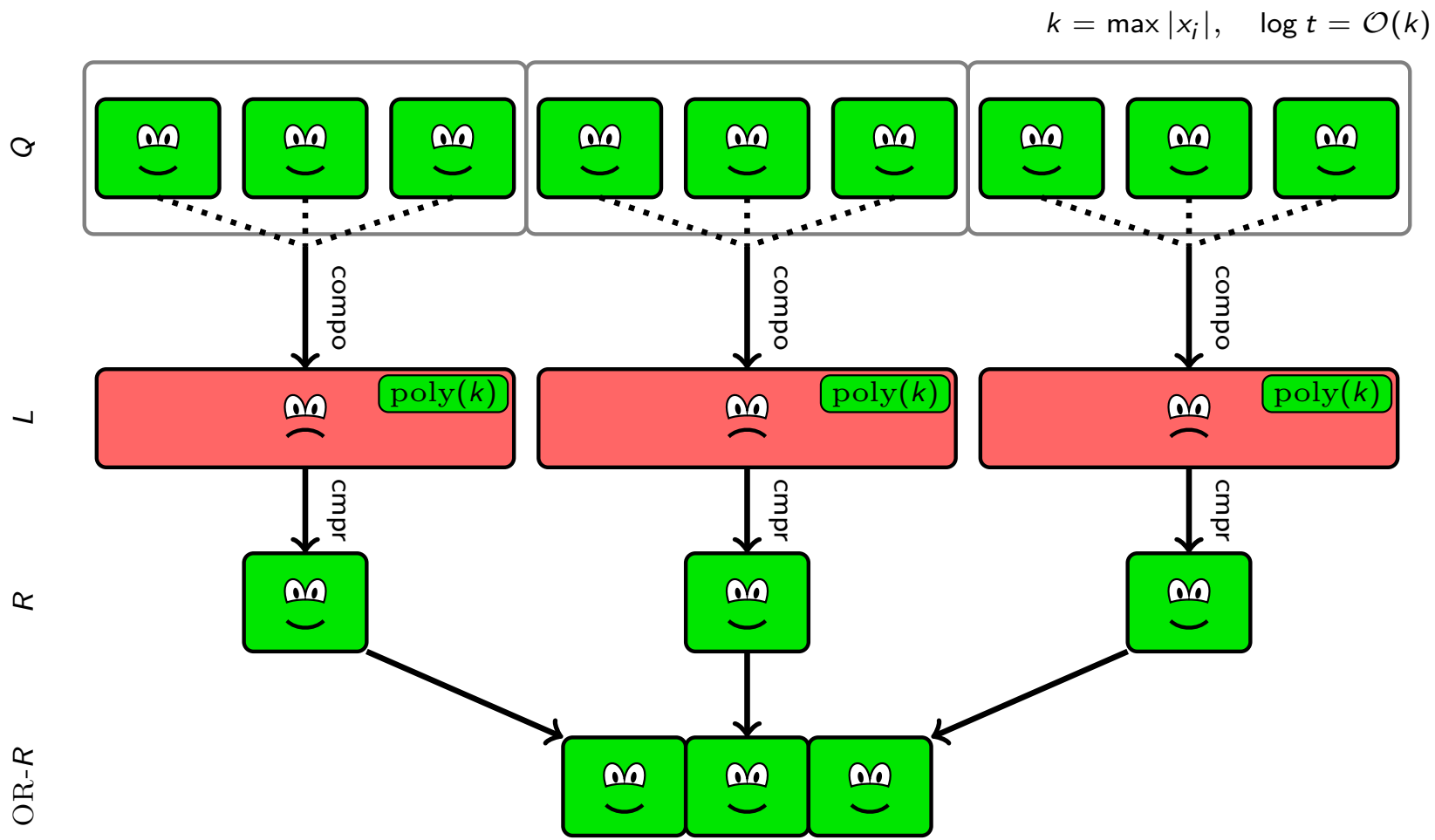
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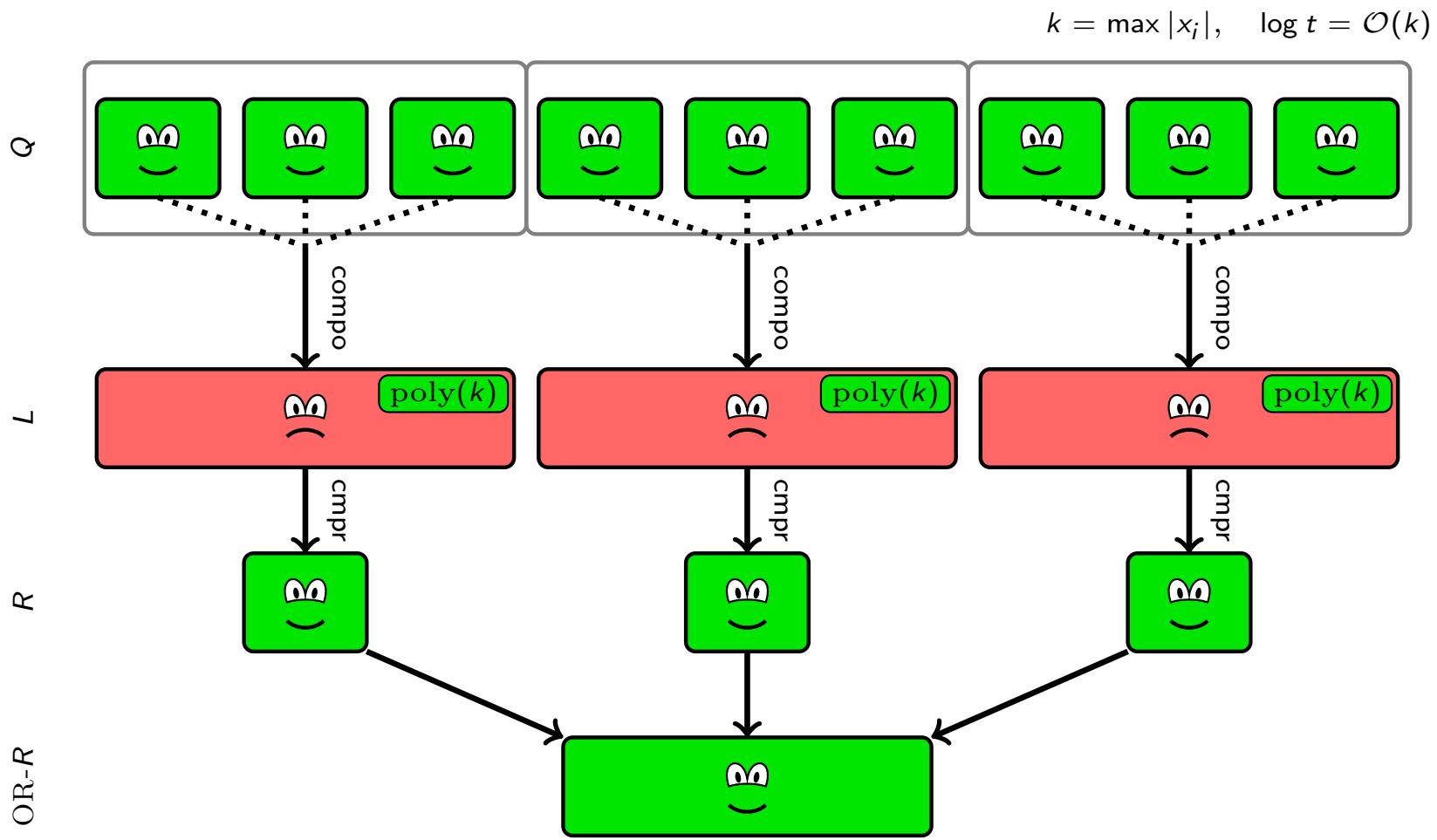
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- In fact, cross-composition is a good framework to express also all the previous results.
- **Plan for now**: show a few cross-compositions and give intuition about basic tricks.

Application 1: Set Splitting

Set Splitting

Input: Universe U and family of subsets $\mathcal{F} \subseteq 2^U$

Parameter: $|U|$

Question: Does there exist a colouring $\mathcal{C} : U \to \{\mathbf{B}, \mathbf{W}\}$ such that every set $X \in \mathcal{F}$ is split, i.e., contains a black and a white element?

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- Assume that t is a power of 2 (by copying the instances).

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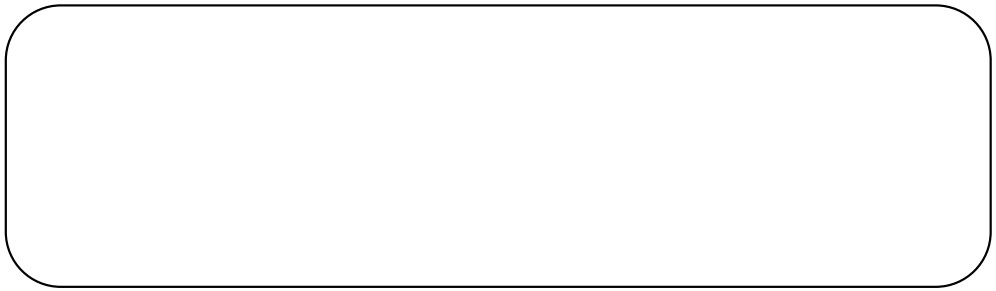

PLAYGROUND

joint universe U

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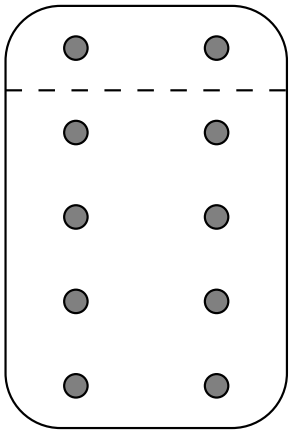
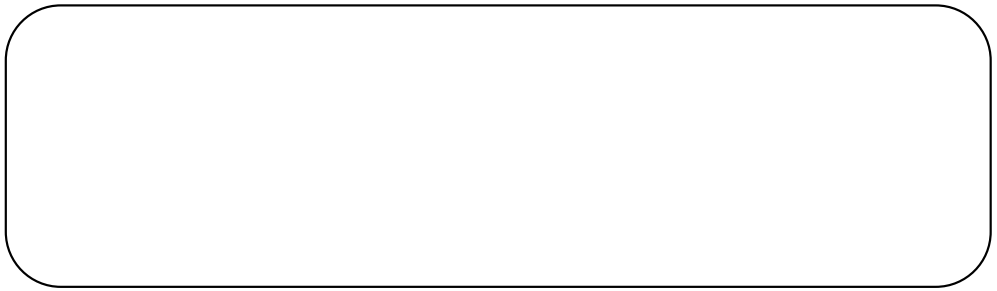
INSTANCE SELECTOR

$1 + \log t$ pairs of vertices





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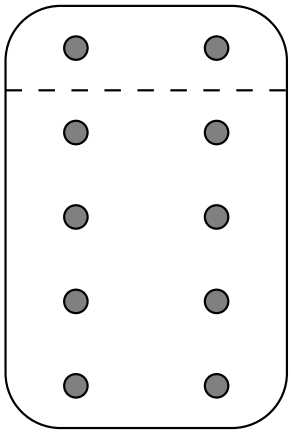

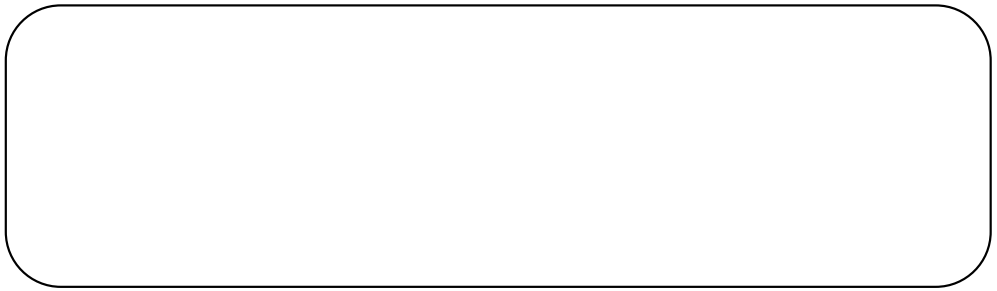

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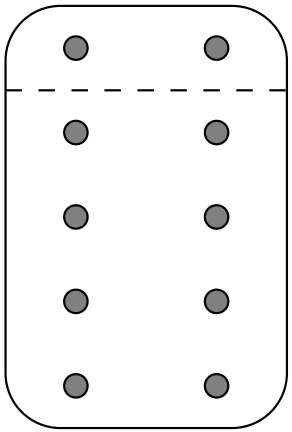

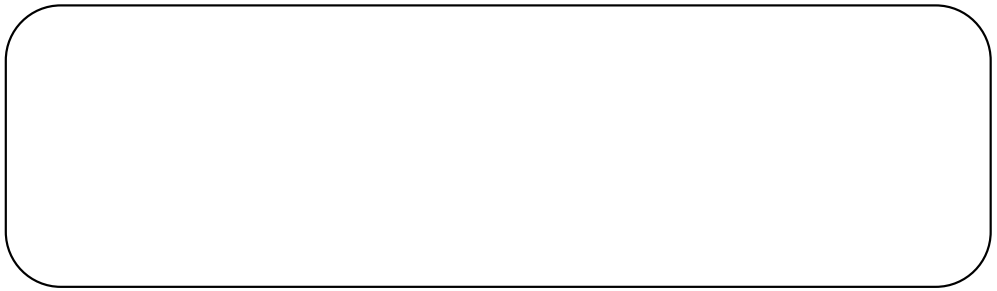

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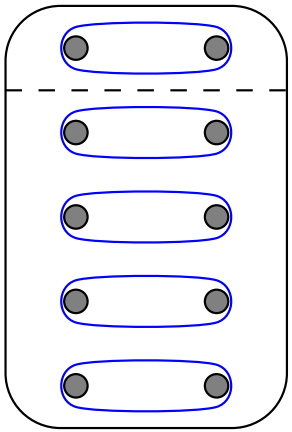

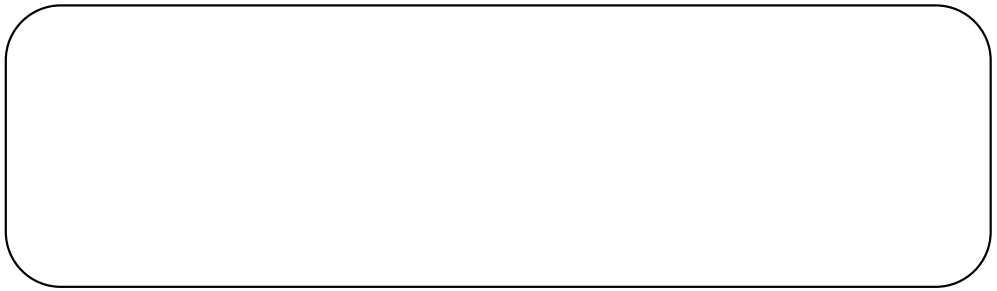

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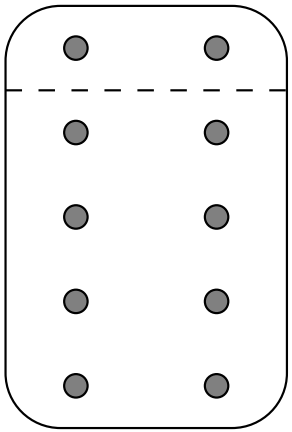

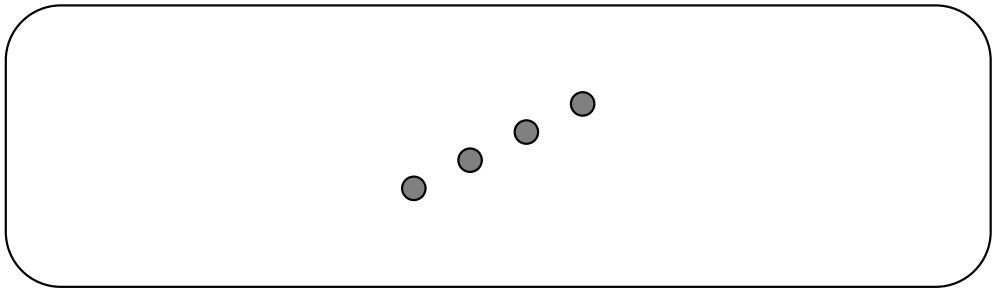

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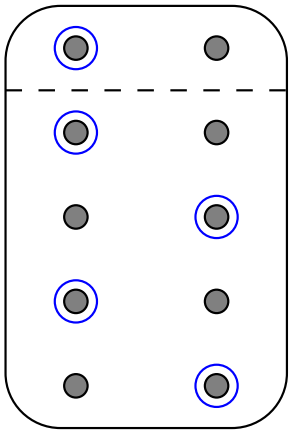
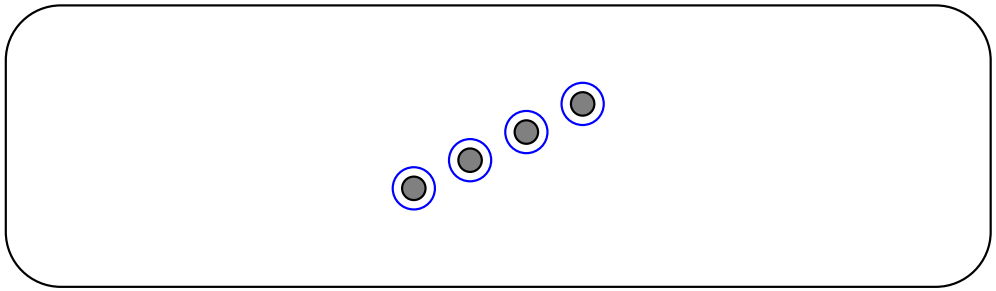


X_0^*: X, left special guy, and binary encoding of i in IS



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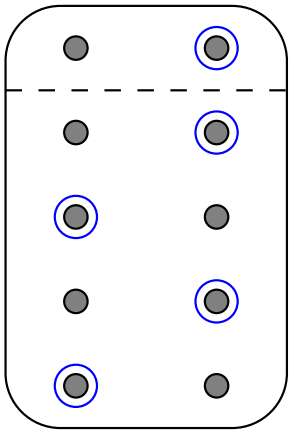

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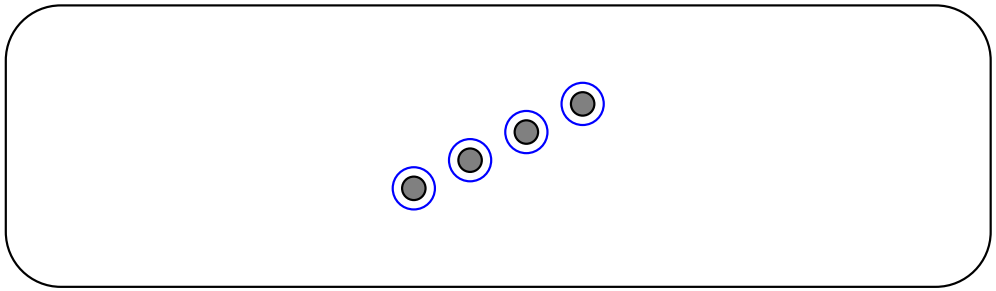

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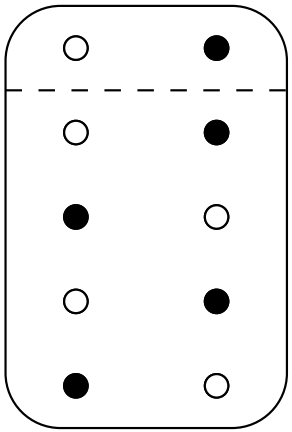

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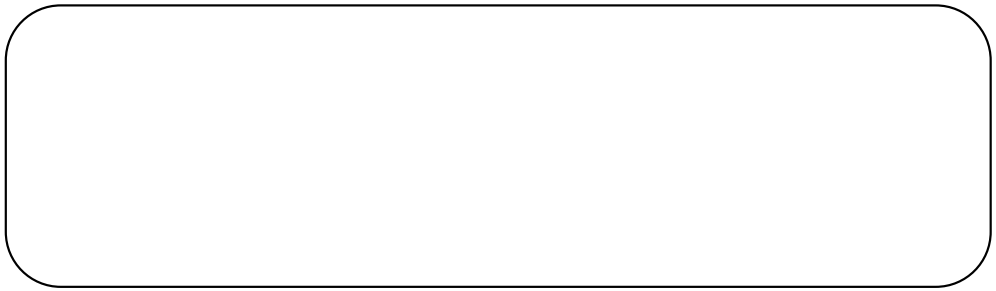

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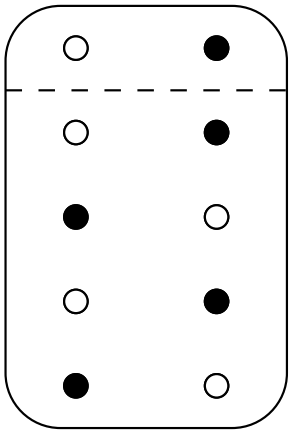

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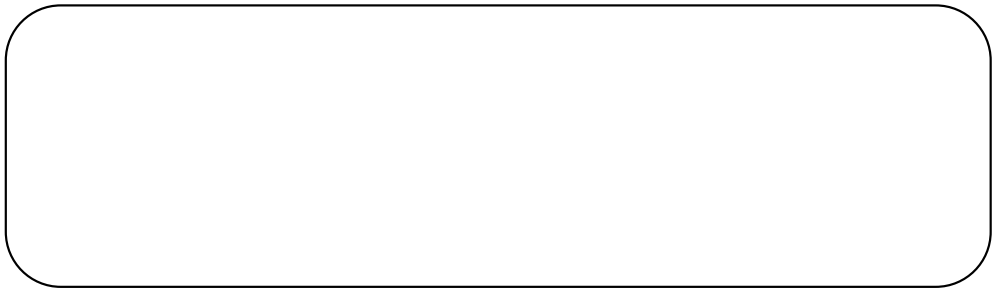
There is exactly one index i with monochromatic parts from IS.



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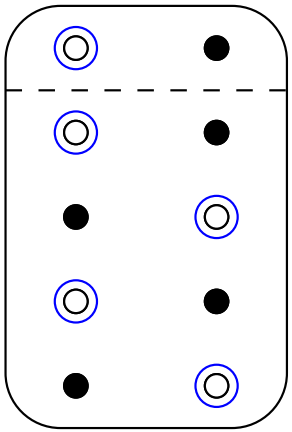

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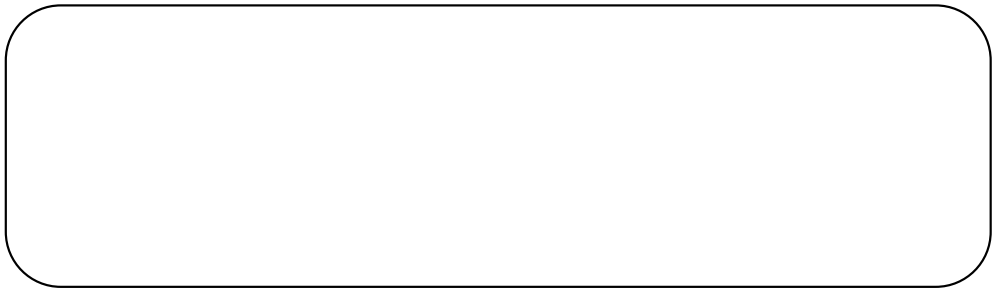

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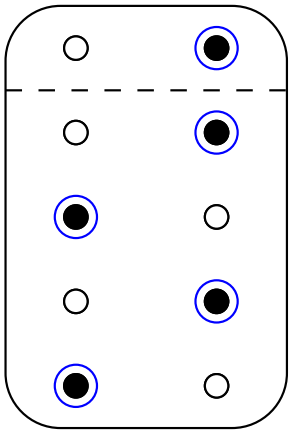

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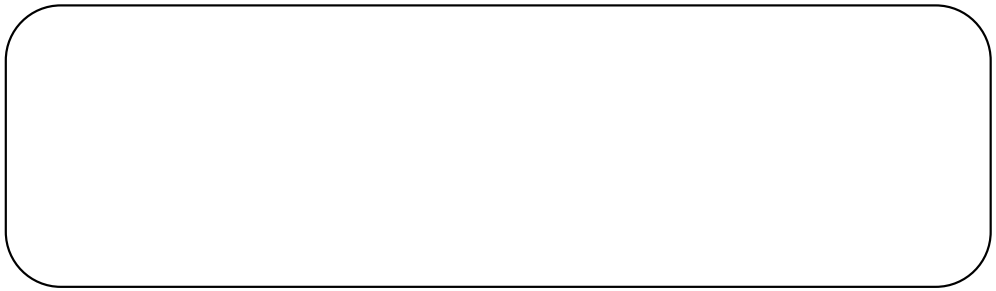

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joint universe U

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INSTANCE SELECTOR

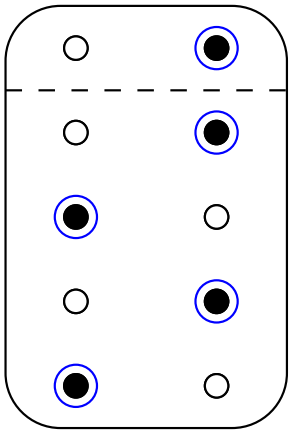
$1 + \log t$ pairs of vertices

Input: Instances $(U, \mathcal{F}^i)$

Output: Instance $(U^*, \mathcal{F}^*)$

$|U^*| = |U| + 2\log t + 2$

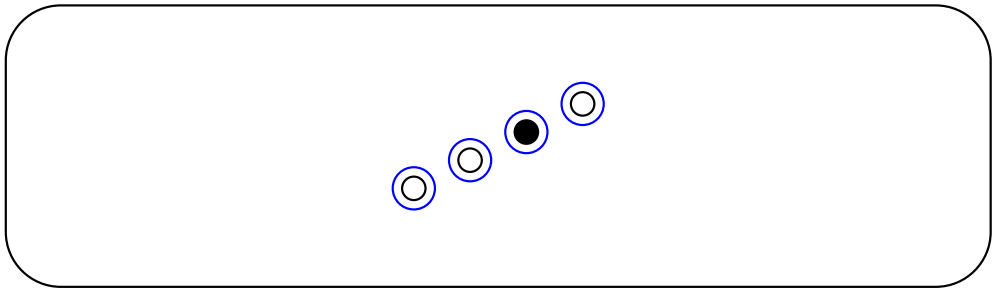
$\mathcal{F}^*$ **consists of**:
$1 + \log t$ 2-element sets for pairs,
$\forall X \in \mathcal{F}^i$, two sets X_0^*, X_1^*



Take any solution $\mathcal{C}$

There is exactly one index i with monochromatic parts from IS.

($\Rightarrow$): $\mathcal{C}$ on IS defines, which instance must be solved in PL



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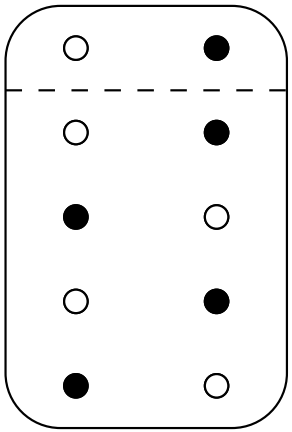

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($\Leftarrow$): If $(U, \mathcal{F}^i)$ is solvable, we set IS accordingly, and solve this instance in PL. Remaining sets are split for free.

PLAYGROUND

joint universe U

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- **Tomorrow**: More combinatorial examples.

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- In particular, Treewidth, Pathwidth, etc. do not admit polykernels, unless **NP** $\subseteq$ **coNP**$/\mathrm{poly}$.

Exercises

Exercise 15.4, points 1, 2, 11, 12, 13.