Lower bounds for polynomial kernelization

Part 2

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Outline

- **Goal**: how to prove that for some problems polynomial kernels do **not** exist?
- **Part 1**:
  - Introduction of the (cross)-composition framework.
  - Basic example.
- **Part 2:**
  - PPT reductions.
  - Case study of several cross-compositions.
  - Weak compositions.

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- Composition+Compression gives OR-Distillation
- OR-Distillation of an **NP**-hard language contradicts **coNP** $\subseteq$ **NP**$/\mathrm{poly}$.
- **Corollary**: To show no-poly-kernel it suffices to construct a composition algorithm.

In the previous episode...

Cross-composition

An unparameterized problem Q *cross-composes* into a parameterized problem L, if there exists a polynomial equivalence relation $\mathcal{R}$ and an algorithm that, given $\mathcal{R}$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

- $(y, k^\star) \in L$ iff $x_i \in Q$ for at least one $i = 1, 2, \ldots, t$,
- $k^\star = \mathrm{poly}\left(\log t + \max_{i=1}^{t} |x_i|\right)$.

Cross-composition theorem

Bodlaender et al.; STACS 2011, SIDMA 2014

If some **NP**-hard problem Q cross-composes into L, then L does not admit a polynomial compression into any language R, unless **NP** $\subseteq$ **coNP**$/\mathrm{poly}$.

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Polynomial parameter transformation (PPT)

A *polynomial parameter transformation* from a parameterized problem P to a parameterized problem Q is a polynomial-time algorithm that transforms a given instance (x, k) of P into an equivalent instance (y, k') of Q such that $k' = \mathrm{poly}(k)$.

PPTs: properties

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If problem P PPT-reduces to Q, and P does not admit a polynomial compression algorithm (into any language R), then neither does Q.

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- **Proof**: Compose the PPT-reduction with the assumed compression for Q.

Application 2: Steiner Tree

Steiner Tree

Input: Graph G with designated terminals $T \subseteq V(G)$, and an integer k

Parameter: $k + |T|$

Question: Is there a set $X \subseteq V(G) \setminus T$, such that $|X| \leq k$ and $G[T \cup X]$ is connected?

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- But we will present an alternative approach.

The pivot problem technique

- Introduce an simpler problem P, which is almost trivially compositional.

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- Move the weight of the proof to the transformation and the actual definition of P.
- **Idea**: Extract the essence of the problem.

Colourful Graph Motif

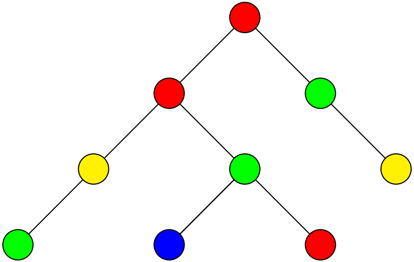
Colourful Graph Motif

Input: Graph G and a colouring function $\mathcal{C} : V(G) \to \{1, 2, \ldots, k\}$

Parameter: k

Question: Does there exists a connected subgraph H of G containing exactly one vertex of each colour?

Colourful Graph Motif — example



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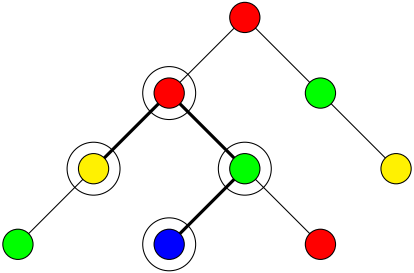

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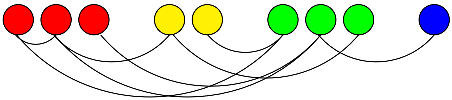
- **Now**: PPT-reduction from CGM to ST.

From CGM to ST



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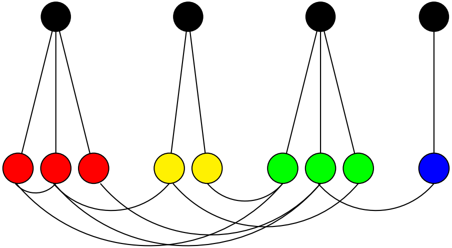

Attach a terminal to every colour class

Give budget k for Steiner nodes

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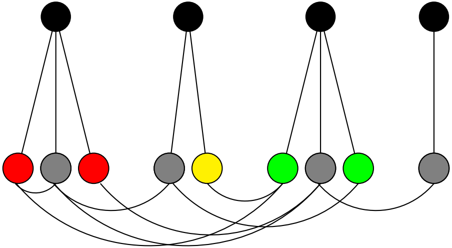

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- CGM PPT-reduces to Steiner Tree par. by $k + |T|$.
- Hence Steiner Tree par. by $k + |T|$ does not admit a polynomial kernel, unless **coNP** $\subseteq$ **NP**$/\mathrm{poly}$.

Application 3: SET COVER par. by $|U|$

SET COVER

Input: Universe U, a family of subsets $\mathcal{F} \subseteq 2^U$, integer k
Parameter: $|U|$
Question: Is there a subfamily $\mathcal{G} \subseteq \mathcal{F}$, $|\mathcal{G}| \leq k$, such that $\bigcup \mathcal{G} = U$?

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- W.l.o.g. $k \leq |U|$.

Colourful Set Cover

Colourful Set Cover

Input: Universe U and families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k \subseteq 2^U$
Parameter: $|U| + k$
Question: Is there a family $\mathcal{G}$ containing exactly one set from each family $\mathcal{F}_i$, such that $\bigcup \mathcal{G} = U$?

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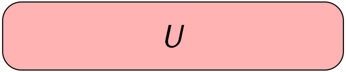
  - Then take $\mathcal{F} = \bigcup \mathcal{F}_i$.
- We will cross-compose Colourful Set Cover into itself.
- **Assumption**: the same universe U, the same k, and t being a power of 2.

Cross-composing into Colourful Set Cover

Input: Instances $(U, (\mathcal{F}^i_j)_{1 \le j \le k})$

Output: Instance $(U^*, (\mathcal{F}^*_j)_{1 \le j \le k})$

Cross-composing into COLOURFUL SET COVER



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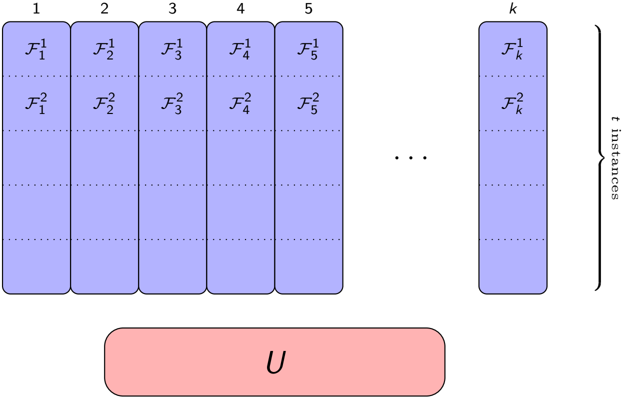

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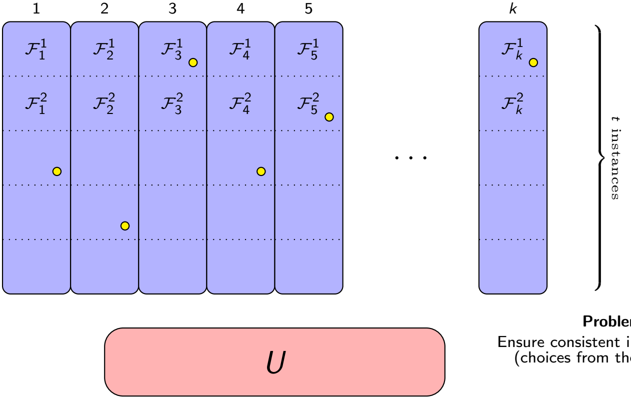

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Ensure consistent instance choice
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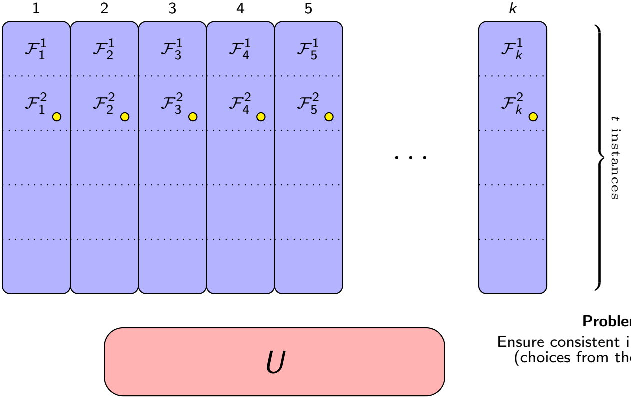

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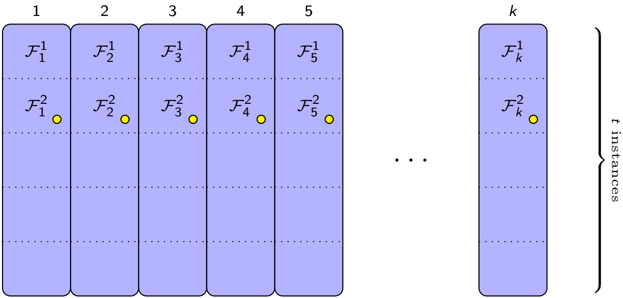

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Equality gadgets

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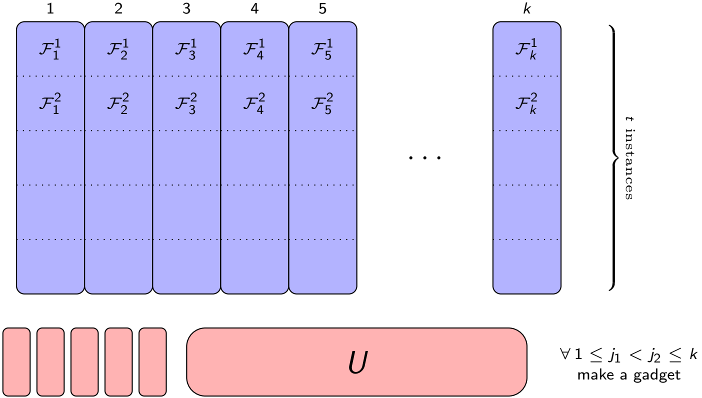

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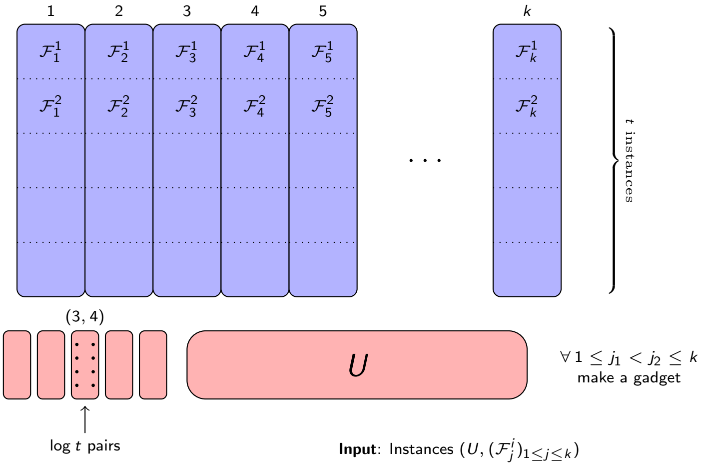

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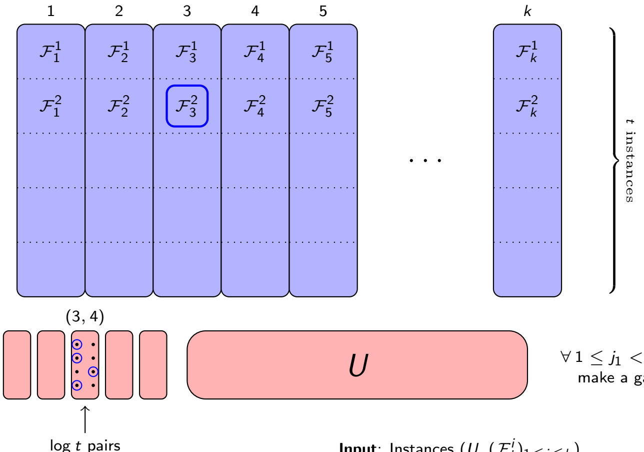

$\forall_i$ add $\mathrm{bin}(i)$ to sets from $\mathcal{F}_3^i$

$\forall\, 1 \le j_1 < j_2 \le k$ make a gadget

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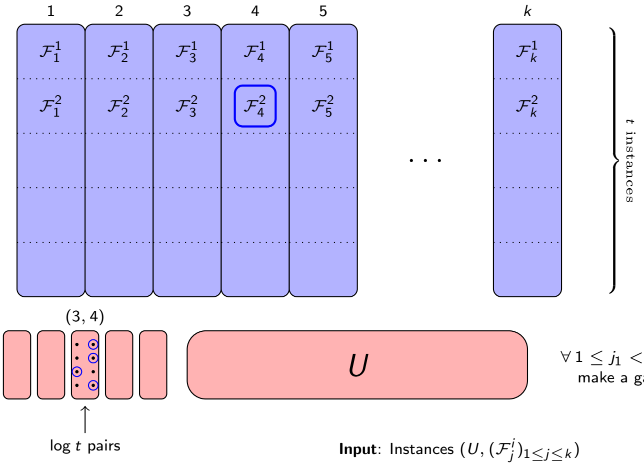

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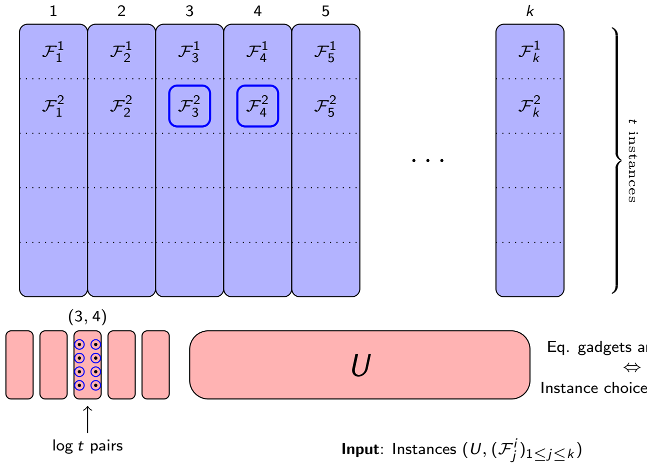

Eq. gadgets are covered
⇔
Instance choices are equal

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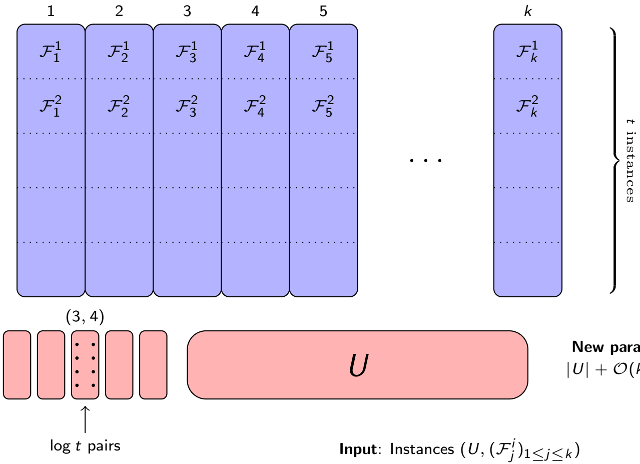

New parameter: $|U| + \mathcal{O}(k^2 \log t)$

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- **Note**: parameterization of Set Cover by $|\mathcal{F}|$ also does not admit a polynomial compression.
  - The composition is quite different.

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- Original motivation of cross-composition.

Application 4: CLIQUE par. by vertex cover

CLIQUE/VC

Input: Graph G, a vertex cover X of G, integer k
Parameter: $|X|$
Question: Is there a clique of size k in G?

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- We make a cross-composition from the standard CLIQUE problem.
- Assume the same number of vertices n and the same target size of the clique k.

Cross-composing into CLIQUE/VC

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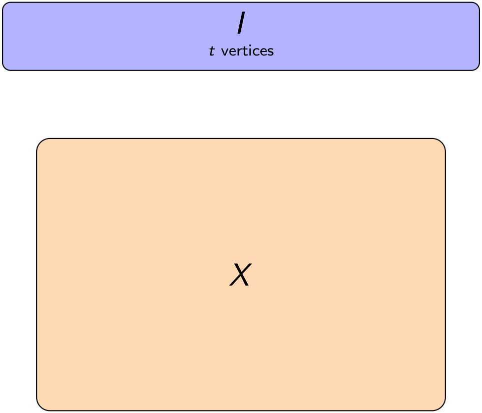
Cross-composing into CLIQUE/VC



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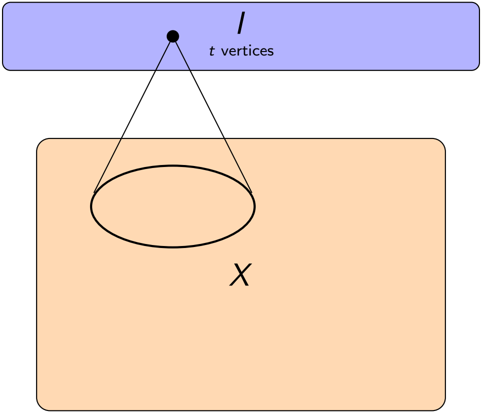

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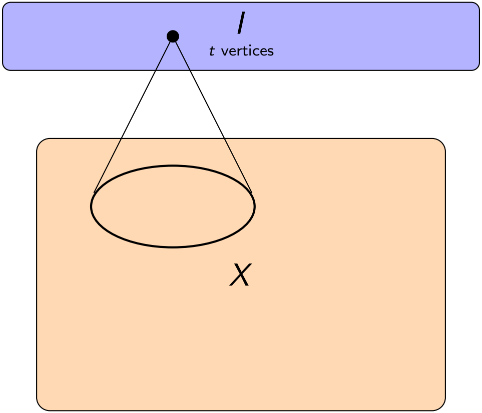

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Problem:
Design a 'universal' modulator X.

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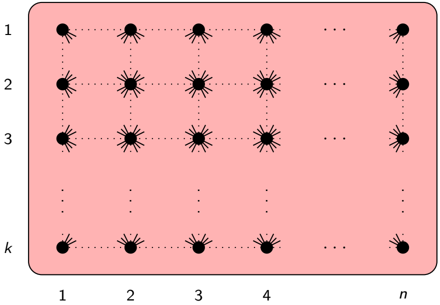

All connections are present except ones in the same row/column.

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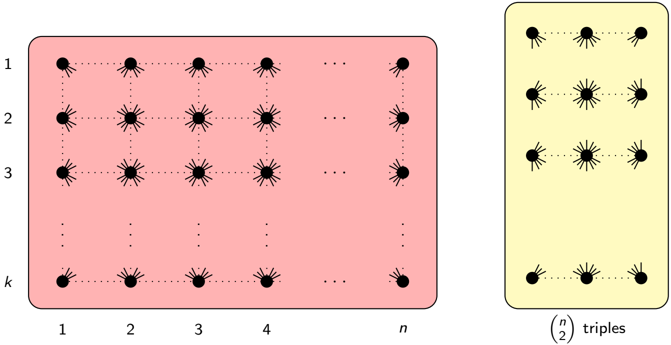

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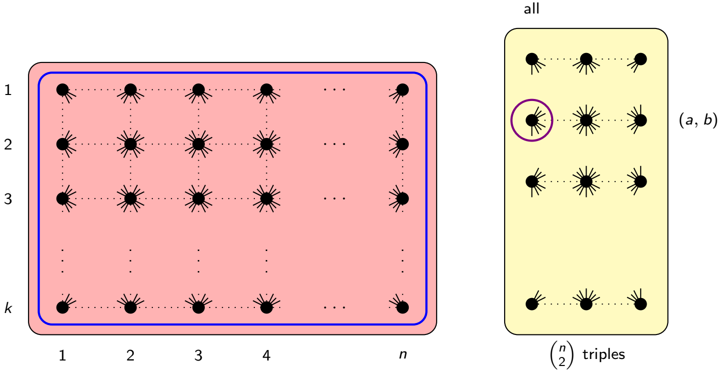

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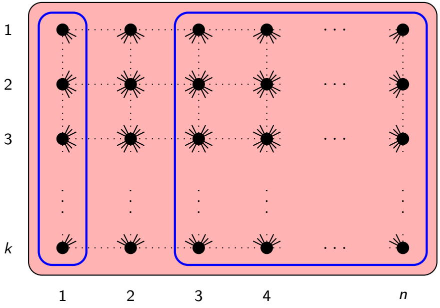

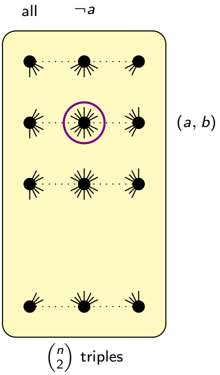

Cross-composing into Clique/VC

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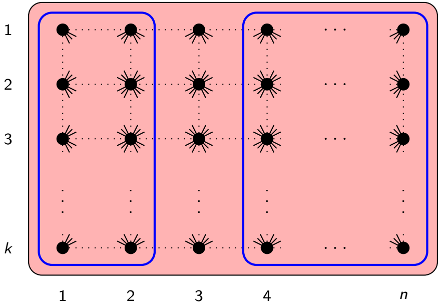

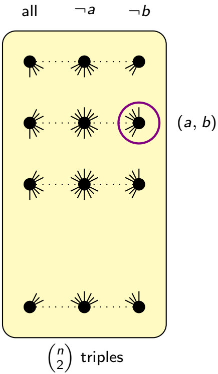

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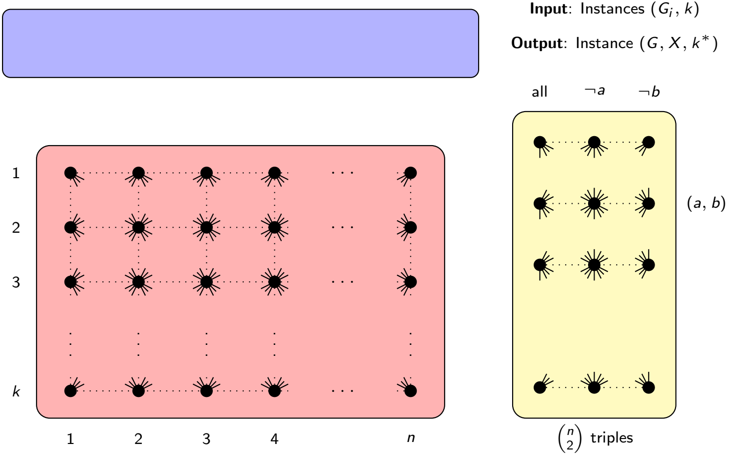

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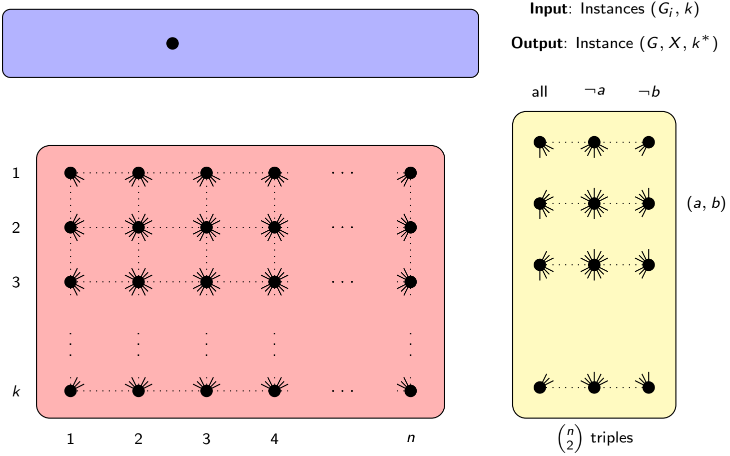

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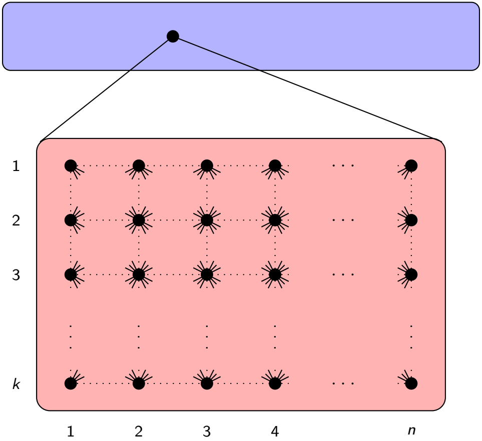

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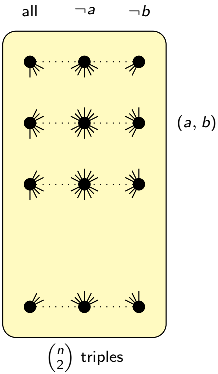

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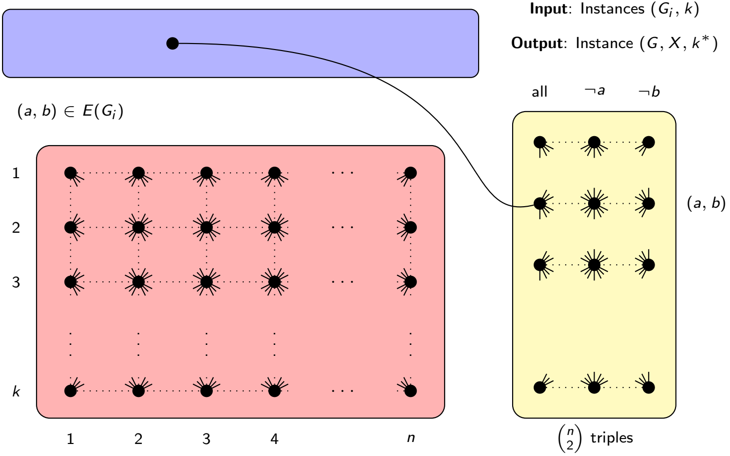

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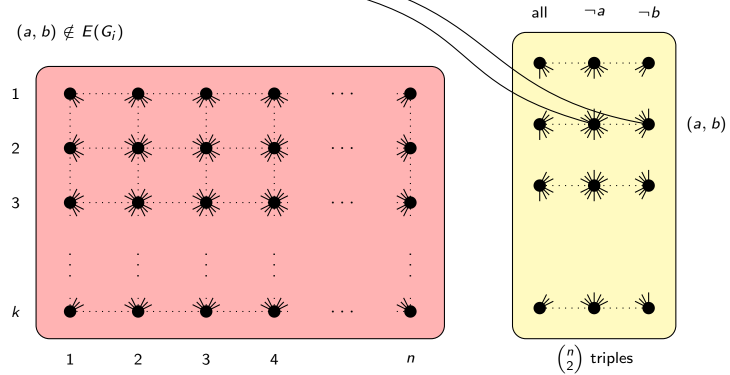
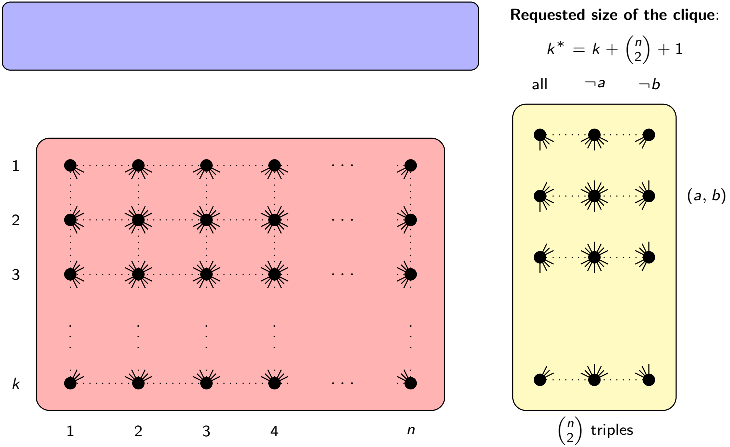


Cross-composing into CLIQUE/VC

Requested size of the clique:

$$k^* = k + \binom{n}{2} + 1$$



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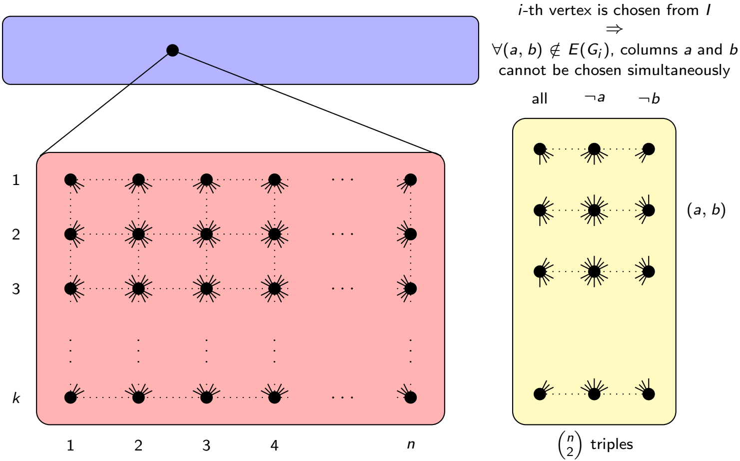

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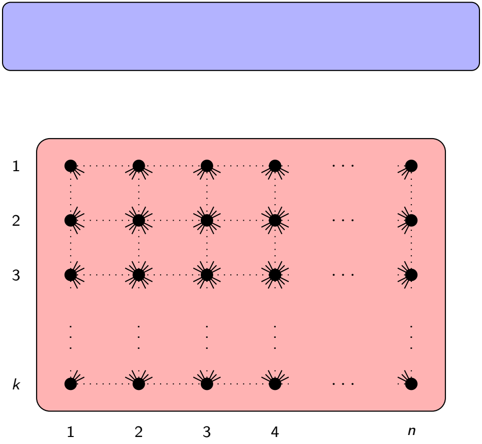
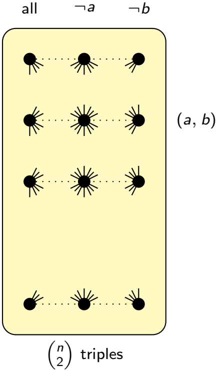


New parameter:

$|X| = kn + 3\binom{n}{2}$



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- *Weak compositions*: proving lower bounds on kernelization complexity for problems that do have polynomial kernels.
- First results by Dell and van Melkebeek (STOC 2010), the framework here by (Dell, Marx; Hermelin, Wu; SODA 2012).

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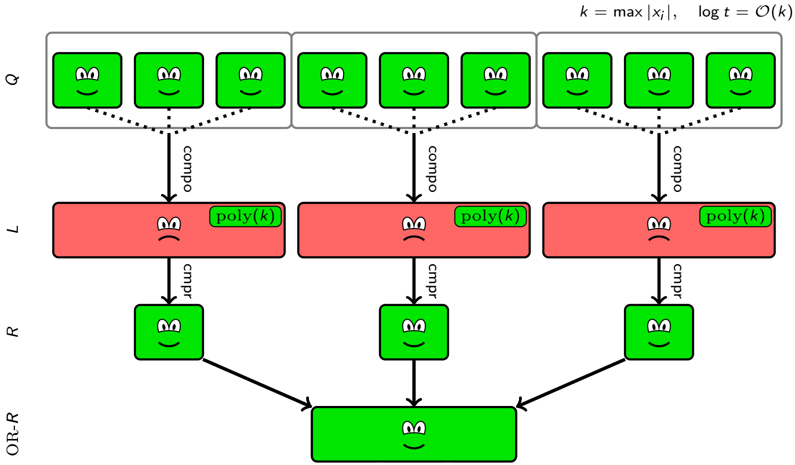
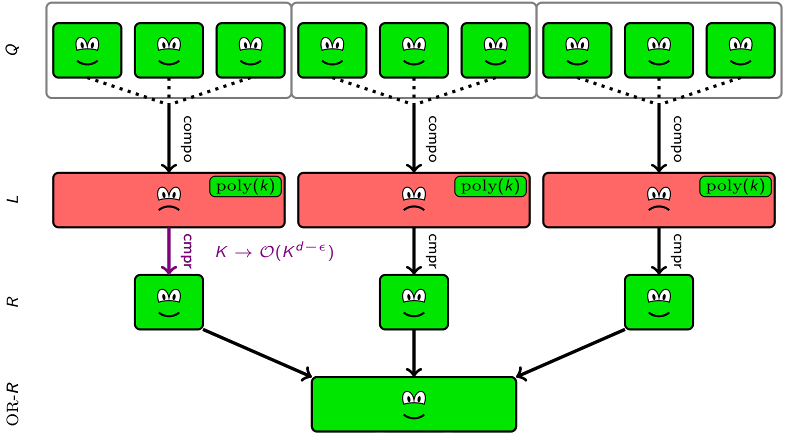
- Let's look again at the proof of the cross-composition Theorem.

Cross-composition proof, recap



Cross-composition proof, recap

$k = \max |x_i|, \quad \log t = \mathcal{O}(k)$



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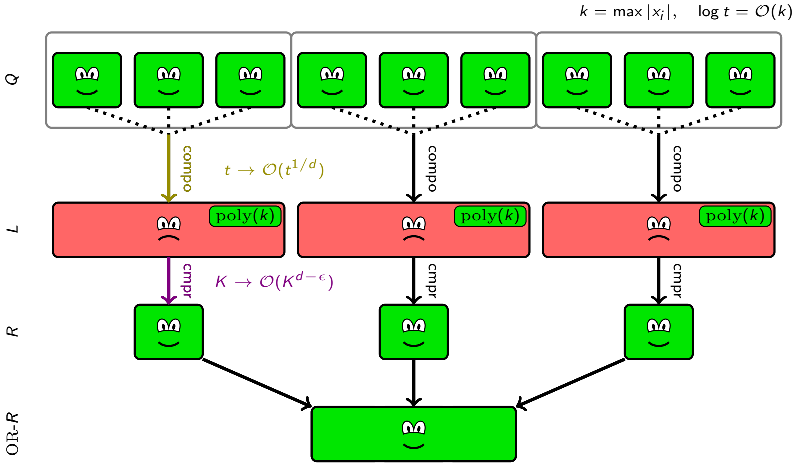

Weak compositions, formally

- It seems that a composition with output bitsize dependence on t being $\mathcal{O}(t^{1/d})$ should exclude compression into bitsize $\mathcal{O}(k^{d-\epsilon})$.

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Weak cross-composition

An unparameterized problem Q *weakly cross-composes* into a parameterized problem L, if there exists a polynomial equivalence relation $\mathcal{R}$, a real constant $d \geq 1$, and an algorithm that, given $\mathcal{R}$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

- $(y, k^\star) \in L$ iff $x_i \in Q$ for at least one $i = 1, 2, \ldots, t$,
- $k^\star = t^{1/d+o(1)} \cdot \mathrm{poly}\left(\max_{i=1}^{t} |x_i|\right)$.

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Suppose some **NP**-hard problem Q admits a weak cross-composition into L with dimension d. Suppose further that L admits a polynomial compression with bitsize $\mathcal{O}(k^{d-\epsilon})$, for some $\epsilon > 0$. Then $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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- **Note**: Also called *cross-composition of bounded cost* by Bodlaender et al. (SIDMA, 2014).

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- Using weak cross-composition we now prove that Vertex Cover does not admit a kernel with bitsize $\mathcal{O}(k^{2-\epsilon})$, for any $\epsilon > 0$. (unless...)

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- **Crux**: choose an appropriate problem Q to start with.

Multicolored Biclique

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Input: Bipartite graph H with bipartition $A \uplus B$, where $A = A_1 \uplus \ldots \uplus A_k$, $B = B_1 \uplus \ldots \uplus B_k$.

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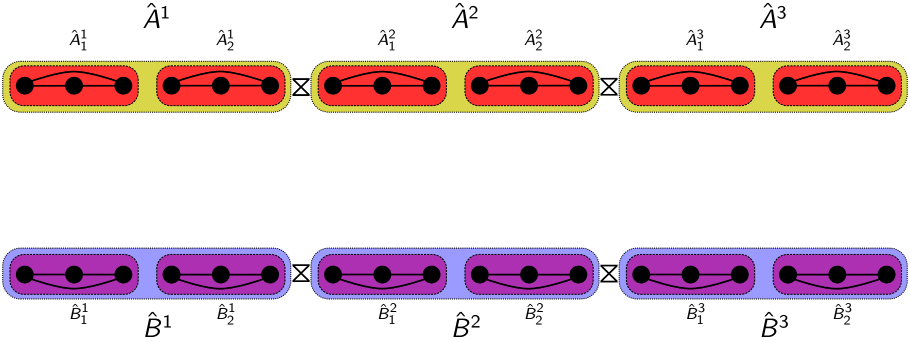
- We provide a weak cross composition of dimension 2 from this version into Vertex Cover.
- **Assumptions**: all the input instances have the same k, each color class has size n in every input instance, $t = s^2$.

Composition

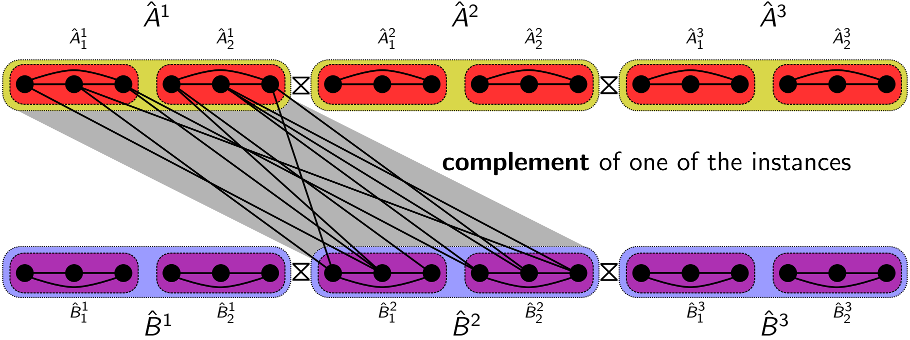
Create s copies of the left side and s copies of the right side.

$N = s \cdot 2kn$



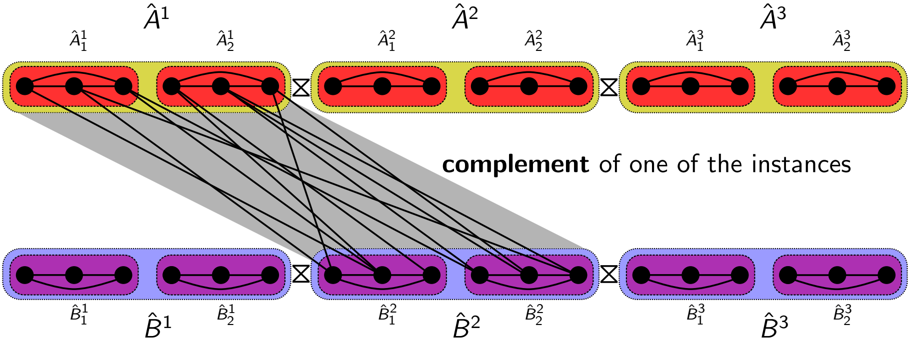
Composition

Embed s^2 instances into s^2 pairs of the sides.



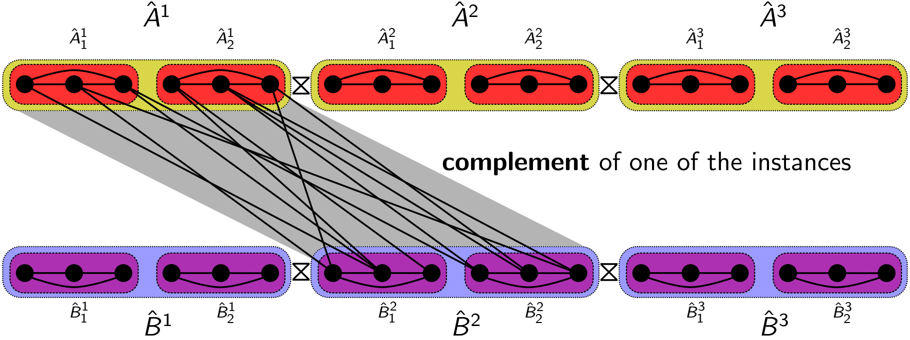
Composition

Ask for an independent set of size $2k$;
equivalently, a vertex cover of size $N - 2k$.



Composition

Edges on the sides ensure choosing one instance and respecting colors.
Edges originating in this instance ensure that the instance is solved.



Wrapping up

- Parameter in the output instance is $N - 2k = t^{1/2} \cdot 2kn - 2k$, which means that the weak composition has dimension 2.

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- Reduction from VC to FVS: add a degree-2 vertex to every edge, thus creating a triangle.
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- For more $\mathcal{O}(k^{d-\epsilon})$ lower bounds, see the book.

Further perspectives

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  - How to show infeasibility of Turing kernelization?

Exercises

Exercise 15.4, all the remaining points.
Exercises 15.1 and 15.5.