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Part III. Minors and planar graphs

SCHOOL ON PARAMETERIZED ALGORITHMS AND COMPLEXITY

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UNIVERSITAS BERGENSIS

Graph Minors

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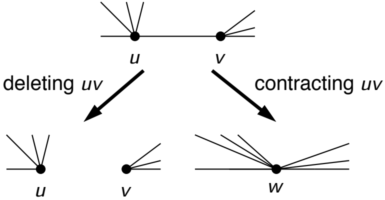
Graph Minors

- Some consequences of the Graph Minors Theorem give a quick way of showing that certain problems are FPT.
- However, the function $f(k)$ in the resulting FPT algorithms can be HUGE, completely impractical.
- History: motivation for FPT.
- Parts and ingredients of the theory are useful for algorithm design.
- New algorithmic results are still being developed.

Graph Minors

Definition: Graph H is a **minor** G $(H \leq G)$ if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

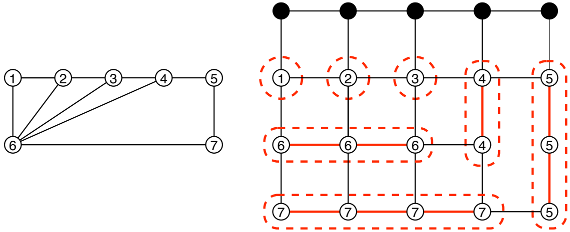


Example: A triangle is a minor of a graph G if and only if G has a cycle (i.e., it is not a forest).

Graph minors

Equivalent definition: Graph H is a **minor** of G if there is a mapping ϕ that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



Minor closed properties

Definition: A set $\mathcal{G}$ of graphs is **minor closed** if whenever $G \in \mathcal{G}$ and $H \leq G$, then $H \in \mathcal{G}$ as well.

Examples of minor closed properties:

planar graphs
acyclic graphs (forests)
graphs having no cycle longer than k
empty graphs

Examples of not minor closed properties:

complete graphs
regular graphs
bipartite graphs

Forbidden minors

Let $\mathcal{G}$ be a minor closed set and let $\mathcal{F}$ be the set of "minimal bad graphs": $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$.

Characterization by forbidden minors:

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\preceq G$$

The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

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Theorem: [Wagner] A graph is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.

In other words: the obstruction set of planarity is $\mathcal{F} = \{K_5, K_{3,3}\}$.

Does every minor closed property have such a finite characterization?

Graph Minors Theorem

Theorem: [Robertson and Seymour] Every minor closed property $\mathcal{G}$ has a finite obstruction set.

Note: The proof is contained in the paper series "Graph Minors I–XX".

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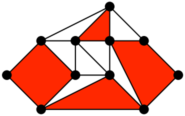
Theorem: [Robertson and Seymour] For every fixed graph H, there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G.

Corollary: For every minor closed property $\mathcal{G}$, there is an $O(n^3)$ time
algorithm for testing whether a given graph G is in $\mathcal{G}$.

Applications

PLANAR FACE COVER: Given a graph G and an integer k, find an embedding of planar graph G such that there are k faces that cover all the vertices.



One line argument:

For every fixed k, the class $\mathcal{G}_k$ of graphs of yes-instances is minor closed.

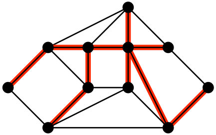
$$\Downarrow$$

For every fixed k, there is a $O(n^3)$ time algorithm for PLANAR FACE COVER.

Note: non-uniform FPT.

Applications

k-Leaf Spanning Tree: Given a graph G and an integer k, find a spanning tree with **at least** k leaves.



Technical modification: Is there such a spanning tree for at least one component of G?

One line argument:

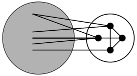
For every fixed k, the class $\mathcal{G}_k$ of no-instances is minor closed.

$$\Downarrow$$

For every fixed k, k-Leaf Spanning Tree can be solved in time $O(n^3)$.

$\mathcal{G} + k$ vertices

Let $\mathcal{G}$ be a graph property, and let $\mathcal{G} + kv$ contain graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.



Lemma: If $\mathcal{G}$ is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k.
$\Rightarrow$ It is (nonuniform) FPT to decide if G can be transformed into a member of $\mathcal{G}$ by deleting k vertices.

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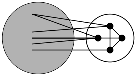

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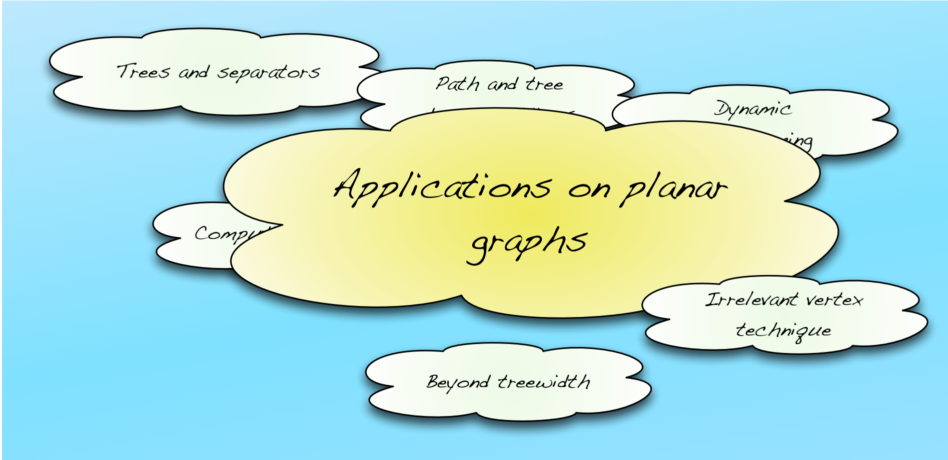
$\Rightarrow$ It is (nonuniform) FPT to decide if G can be transformed into a member of $\mathcal{G}$ by deleting k vertices.

- If $\mathcal{G}$ = forests $\Rightarrow$ $\mathcal{G} + kv$ = graphs that can be made acyclic by the deletion of k vertices $\Rightarrow$ Feedback Vertex Set is FPT.
- If $\mathcal{G}$ = planar graphs $\Rightarrow$ $\mathcal{G} + kv$ = graphs that can be made planar by the deletion of k vertices (k-apex graphs) $\Rightarrow$ k-Apex Graph is FPT.
- If $\mathcal{G}$ = empty graphs $\Rightarrow$ $\mathcal{G} + kv$ = graphs with vertex cover number at most k $\Rightarrow$ Vertex Cover is FPT.


Trees and separators
Path and tree
Dynamic
Applications on planar graphs
Irrelevant vertex technique
Beyond treewidth

Recap: Tree decomposition

A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \chi)$, where T is a tree and mapping χ assigns to every node t of T a vertex subset X_t (called a bag) such that

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A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \chi)$, where T is a tree and mapping χ assigns to every node t of T a vertex subset X_t (called a bag) such that

(T1) $\bigcup_{t \in V(T)} X_t = V(G)$.

(T2) For every $vw \in E(G)$, there exists a node t of T such that bag $\chi(t) = X_t$ contains both v and w.

(T3) For every $v \in V(G)$, the set $\chi^{-1}(v)$, i.e. the set of nodes $T_v = \{t \in V(T) \mid v \in X_t\}$ forms a connected subgraph (subtree) of T.

The *width* of tree decomposition $\mathcal{T} = (T, \chi)$ equals $\max_{t \in V(T)} |X_t| - 1$, i.e the maximum size of its bag minus one. The *treewidth* of a graph G is the minimum width of a tree decomposition of G.

Applications of treewidth

In parameterized algorithms various modifications of WIN/WIN approach: either treewidth is small, and we solve the problem, or something good happens

- Finding a path of length $\geq k$ is FPT because every graph with treewidth k contains a k-path

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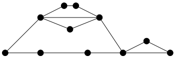
- Finding a path of length $\geq k$ is FPT because every graph with treewidth k contains a k-path
- Feedback vertex set is FPT because if the treewidth is more than k, the answer is NO.
- Disjoint Path problem is FPT because if the treewidth is $\geq f(k)$, then the graph contains irrelevant vertex (non-trivial arguments)

Properties of treewidth

Fact: treewidth ≤ 2 if and only if graph is subgraph of a series-parallel graph



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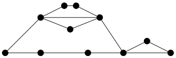
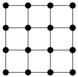


Fact: For every $k \geq 2$, the treewidth of the $k \times k$ grid is exactly k.



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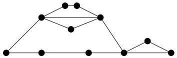

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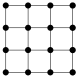

Fact: Treewidth does not increase if we delete edges, delete vertices, or contract edges.

$\Longrightarrow$ If F is a minor of G, then the treewidth of F is at most the treewidth of G.

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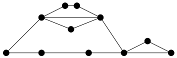

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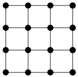

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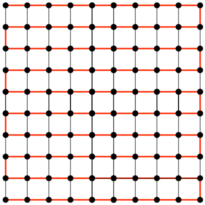
The treewidth of the k-clique is $k - 1$.

Obstruction to Treewidth

Another, extremely useful, obstructions to small treewidth are grid-minors. Let t be a positive integer. The $t \times t$-*grid* $\boxplus_t$ is a graph with vertex set $\{(x, y) \mid x, y \in \{1, 2, \ldots, t\}\}$. Thus $\boxplus_t$ has exactly t^2 vertices. Two different vertices (x, y) and (x', y') are adjacent if and only if $|x - x'| + |y - y'| \leq 1$.



If a graph contains large grid as a minor, its treewidth is also large.

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What is much more surprising, is that the converse is also true: every graph of large treewidth contains a large grid as a minor.

Theorem (Excluded Grid Theorem, Robertson, Seymour and Thomas, 1994)

If the treewidth of G *is at least* $k^{4t^2(t+2)}$*, then* G *has* $\boxplus_t$ *as a minor.*

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Theorem (Excluded Grid Theorem, Chekuri and Chuzhoy, 2013)

Let $t \geq 0$ be an integer. There exists a universal constant c, such that every graph of treewidth at least $c \cdot t^{99}$ contains $\boxplus_t$ as a minor.

Excluded Grid Theorem A : Planar Graph

Our set of treewidth applications is based on the following

Theorem (Planar Excluded Grid Theorem, Robertson, Seymour and Thomas; Guo and Tamaki)

Let $t \geq 0$ be an integer. Every planar graph G of treewidth at least $\frac{9}{2}t$, contains $\boxplus_t$ as a minor. Furthermore, there exists a polynomial-time algorithm that for a given planar graph G either outputs a tree decomposition of G of width $\frac{9}{2}t$ or constructs a minor model of $\boxplus_t$ in G.

Grid Theorem: Sketch of the proof

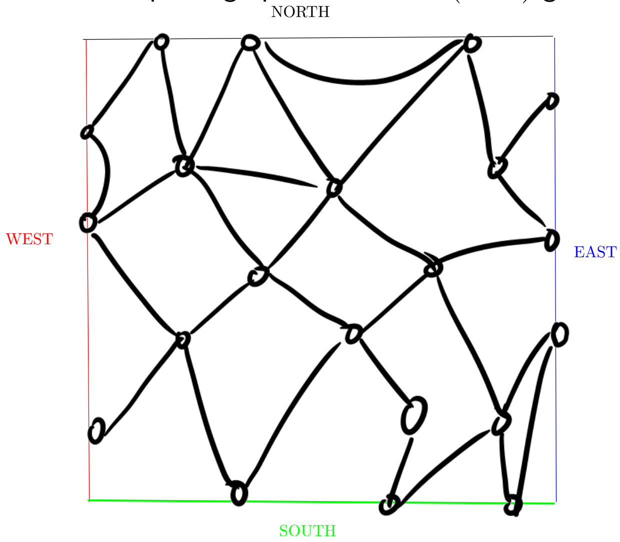
The proof is based on Menger's Theorem

Theorem (Menger 1927)

Let G be a finite undirected graph and x and y two nonadjacent vertices. The size of the minimum vertex cut for x and y (the minimum number of vertices whose removal disconnects x and y) is equal to the maximum number of pairwise vertex-disjoint paths from x to y.

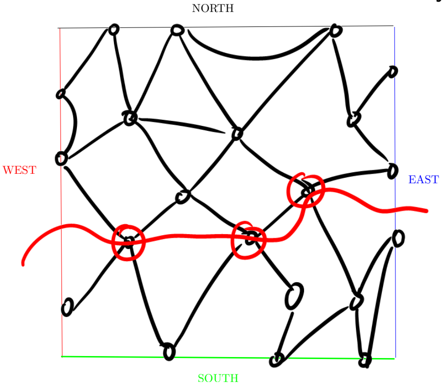
Grid Theorem: Sketch of the proof

Let G be a plane graph that has no $(\ell \times \ell)$-grid as a minor.



Grid Theorem: Sketch of the proof

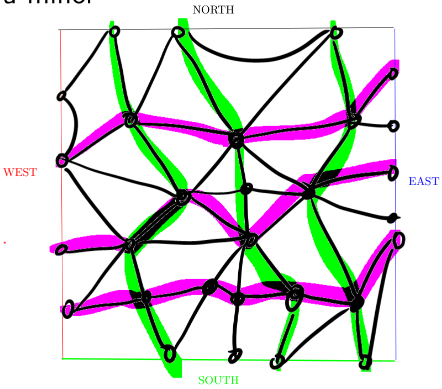
Either East can be separated from West, or South from North by



removing at most ℓ vertices

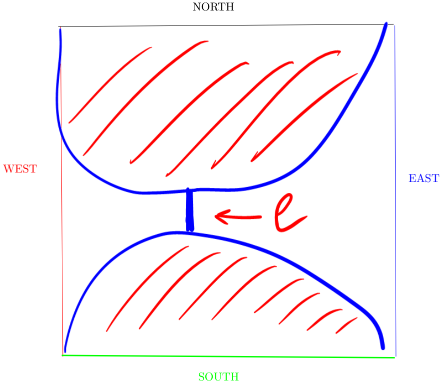
Grid Theorem: Sketch of the proof

Otherwise by making use of Menger we can construct $\ell \times \ell$ grid as a minor



Grid Theorem: Sketch of the proof

Partition the edges. Every time the middle set contains only vertices of East, West, South, and North, at most 4ℓ in total.



Grid Theorem: Sketch of the proof

"At this point we have reached a degree of handwaving so exuberant, one may fear we are about to fly away. Surprisingly, this handwaving has a completely formal theorem behind it."

(Ryan Williams 2011, SIGACT News)

Excluded Grid Theorem: Planar Graphs

One more Excluded Grid Theorem, this time not for minors but just for edge contractions.

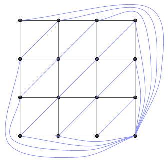


Figure : A triangulated grid Γ_4.

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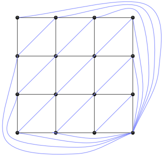

Figure : A triangulated grid Γ_4.

For an integer $t > 0$ the graph Γ_t is obtained from the grid $\boxplus_t$ by adding for every $1 \leq x, y \leq t-1$, the edge $(x,y),(x+1,y+1)$, and making the vertex (t,t) adjacent to all vertices with $x \in \{1,t\}$ and $y \in \{1,t\}$.

Excluded Grid Theorem: Planar Graphs

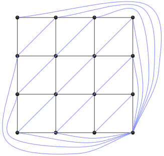

Figure : A triangulated grid Γ_4.

Theorem

For any connected planar graph G and integer $t \geq 0$, if **tw**$(G) \geq 9(t+1)$*, then G contains Γ_t as a contraction. Furthermore there exists a polynomial-time algorithm that given G either outputs a tree decomposition of G of width $9(t+1)$ or a set of edges whose contraction result in Γ_t.*

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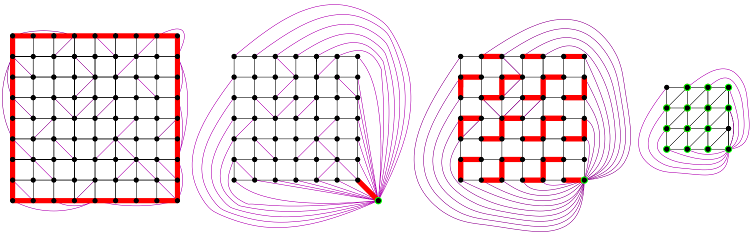
Proof sketch



Shifting Techniques

Locally bounded treewidth

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Let G be a planar graph, $v \in V(G)$ and $r \geq 1$. Then $\mathbf{tw}(G_v^r) \leq 18(r+1)$.

Proof.

Hint: use contraction-grid theorem. □

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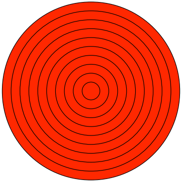
$18(r+1)$ in the above lemma can be made $3r+1$.

Locally bounded treewidth

Lemma

Let v be a vertex of a planar graph G and let L_i, be the vertices of G at distance i, $0 \leq i \leq n$, from v. Then for any $i, j \geq 0$, the treewidth of the subgraph $G_{i,i+j}$ induced by vertices in $L_i \cup L_{i+1} \cup \cdots \cup L_{i+j}$ does not exceed $3j + 1$.

Proof.



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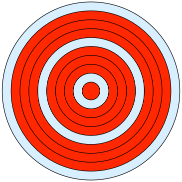

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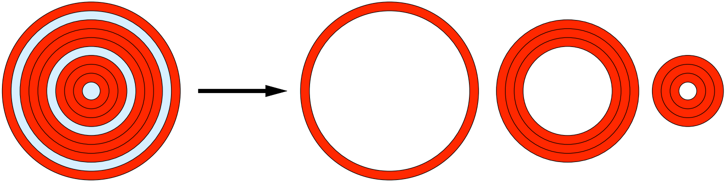

□

Intuition

The idea behind the shifting technique is as follows:

- Pick a vertex v of planar graph G and run breadth-first search (BFS) from v.
- For any $i, j \geq 0$, the treewidth of the subgraph $G_{i,i+j}$ induced by vertices in levels $i, i+1, \ldots, i+j$ of BFS does not exceed $3j+1$.
- Now for an appropriate choice of parameters, we can find a "shift" of "windows", i.e. a disjoint set of a small number of consecutive levels of BFS, "covering" the solution. Because every window is of small treewidth, we can employ the dynamic programing or the power of Courcelle's theorem to solve the problem.

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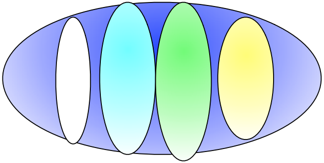
We will see two examples.

Useful viewpoint

Lemma

Let G be a planar graph and k be an integer, $1 \geq k \leq |V(G)|$. Then the vertex set of G can be partitioned into k sets such that any $k-1$ of the sets induce a graph of treewidth at most $3k-2$. Moreover, such a partition can be found in polynomial time.



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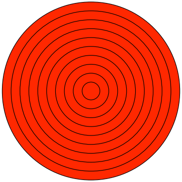

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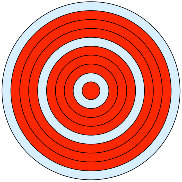

□

Useful viewpoint

Lemma

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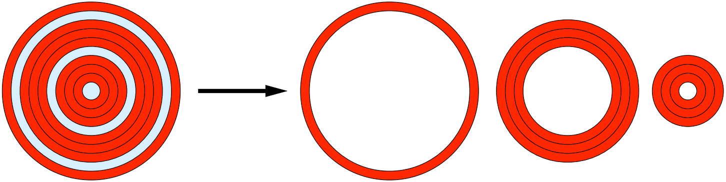

□

Example 1: Subgraph Isomorphism

Subgraph Isomorphism: given graphs H and G, find a copy of H in G as subgraph. Parameter $k := |V(H)|$.

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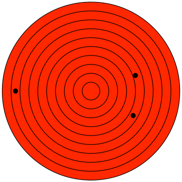
$\mathbf{MSO}_2$ formula of size $k^{O(1)}$ for Subgraph Isomorphism exists.

Courcelle's Theorem implies that we have $f(k,t) \cdot n$ time algorithm for Subgraph Isomorphism on graphs of treewidth t.

Example 1: Subgraph Isomorphism

- Partition the vertex set of G into $k+1$ sets $S_0 \cup \cdots \cup S_k$ such that for every $i \in \{0, \ldots, k\}$, graph $G - S_i$ is of treewidth at most $3k+1$.



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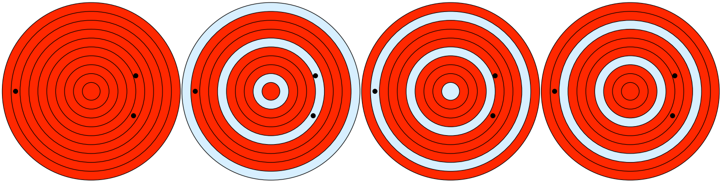
- For every k-vertex subset X of G, there is $i \in \{0, \ldots, k\}$ such that $X \cap S_i = \emptyset$. Therefore, if G contains H as a subgraph, then for at least one value of i, $G - S_i$ also contains H.



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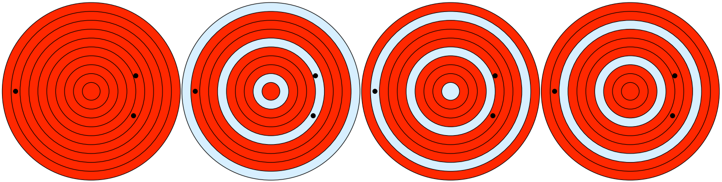
- For every k-vertex subset X of G, there is $i \in \{0, \ldots, k\}$ such that $X \cap S_i = \emptyset$. Therefore, if G contains H as a subgraph, then for at least one value of i, $G - S_i$ also contains H.
- It means that by trying each of the graphs $G - S_i$ for each $i \in \{0, \ldots, k\}$, we find a copy of H in G if there is one.



Example 1: Subgraph Isomorphism

Theorem

Subgraph Isomorphism on planar graphs is FPT parameterized by $|V(H)|$.

Example 2: Bisection

For a given n-vertex graph G, weight function $w : V(G) \to \mathbb{N}$ and integer k, the task is to decide if there is a partition of $V(G)$ into sets V_1 and V_2 of weights $\lceil w(V(G))/2 \rceil$ and $\lfloor w(V(G)/2 \rfloor$ and such that the number of edges between V_1 and V_2 is at most k. In other words, we are looking for a balanced partition (V_1, V_2) with a (V_1, V_2)-cut of size at most k.

Example 2: Bisection. Building blocks.

Lemma

BISECTION is solvable in time $2^t \cdot n^{\mathcal{O}(1)}$ on an n-vertex given together with its tree decomposition of width t.

Lemma

Let G be a planar graph and k be an integer, $1 \geq k \leq |E(G)|$. Then the edge set of G can be partitioned into k sets such that after contracting edges of any of these sets, the treewidth of the resulting graph does not exceed ck for some constant $c > 0$. Moreover, such a partition can be found in polynomial time.

Proof.

Grid theorem, what else? On board. □

Example 2: Bisection

Theorem

BISECTION *on planar graphs is solvable in time* $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$*.*

Proof.

□

Shifting technique: history

- Originated as a tool for obtaining PTAS. The basic idea due to Baker (1994)
- Eppstein: the notion of local treewidth (1995)
- Grohe: extending to H-minor-free graphs (2003)
- Demaine, Hajiaghayi, and Kawarabayashi contractions on H-minor-free graphs (2005).

Bidimensionality

Bidimensionality

Subexponential algorithms, EPTAS, kernels on planar, bounded genus, H-minor free graphs...

Reminder: Grid Theorem

Theorem (Planar Excluded Grid Theorem)

Let $t \geq 0$ be an integer. Every planar graph G of treewidth at least $\frac{9}{2}t$, contains $\boxplus_t$ as a minor. Furthermore, there exists a polynomial-time algorithm that for a given planar graph G either outputs a tree decomposition of G of width $\frac{9}{2}t$ or constructs a minor model of $\boxplus_t$ in G.

Lipton-Tarjan Theorem

Corollary

The treewidth of an n-vertex planar graph is $\mathcal{O}(\sqrt{n})$

Vertex Cover on planar graphs. Just three questions

Does a planar graph contains a vertex cover of size at most k?

- VERTEX COVER has a kernel with at most $2k$ vertices which is an induced subgraph of the input graph. Thus when the input graph is planar we obtain in polynomial time an equivalent planar instance of size at most $2k$.

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- VERTEX COVER has a kernel with at most $2k$ vertices which is an induced subgraph of the input graph. Thus when the input graph is planar we obtain in polynomial time an equivalent planar instance of size at most $2k$.
- Find a tree decomposition
- Dynamic programming solves VERTEX COVER in time $2^{\mathcal{O}(\sqrt{t})}n^{\mathcal{O}(1)} = 2^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$

Other problems on Planar Graphs

What about other problems like INDEPENDENT SET, FEEDBACK VERTEX SET, DOMINATING SET or k-PATH?

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What about other problems like INDEPENDENT SET, FEEDBACK VERTEX SET, DOMINATING SET or k-PATH?

- For most of the problems, obtaining a kernel is not that easy, and
- For some like k-PATH, we know that no polynomial kernel exists (of course unless ....)

Vertex Cover, one more try

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(i) Compute the treewidth of G. If it is more than $c\sqrt{k}$—say NO. (It contains $\boxplus_{2\sqrt{k}}$ as a minor...)

(ii) If the treewidth is less than $c\sqrt{k}$, do DP.

What is special in Vertex Cover?

Same strategy should work for any problem if

- $(P1)$ The size of any solution in $\boxplus_t$ is of order $\Omega(t^2)$.
- $(P2)$ On graphs of treewidth t, the problem is solvable in time $2^{\mathcal{O}(t)} \cdot n^{\mathcal{O}(1)}$.
- $(P3)$ The problem is minor-closed, i.e. if G has a solution of size k, then every minor of G also has a solution of size k.

This settles FEEDBACK VERTEX SET and k-PATH. Why not DOMINATING SET?

Reminder: Contracting to a grid

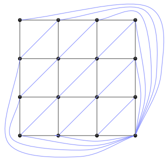

Figure : A triangulated grid Γ_4.

Theorem

For any connected planar graph G and integer $t \geq 0$, if $\mathbf{tw}(G) \geq 9(t+1)$*, then G contains Γ_t as a contraction. Furthermore there exists a polynomial-time algorithm that given G either outputs a tree decomposition of G of width $9(t+1)$ or a set of edges whose contraction result in Γ_t.*

Strategy for Dominating Set

Same strategy should work for any problem with:

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This settles DOMINATING SET

Lets try to formalize

Restrict to vertex-subset problems.
Let ϕ be a computable function which takes as an input graph G, a set $S \subseteq V(G)$ and outputs **true** or **false**.
For an example, for Dominating Set: $\phi(G, S) = \mathbf{true}$ if and only if $N[S] = V(G)$.

Lets try to formalize

Definition

For function ϕ, we define *vertex-subset problem* Π as a parameterized problem, where input is a graph G and an integer k, the parameter is k.

For *maximization* problem, the task is to decide whether there is a set $S \subseteq V(G)$ such that $|S| \geq k$ and $\phi(G, S) = \mathbf{true}$.
Similarly, for *minimization* problem, we are looking for a set $S \subseteq V(G)$ such that $|S| \leq k$ and $\phi(G, S) = \mathbf{true}$.

Optimization problem

Definition

For a vertex-subset minimization problem Π,

$$OPT_{\Pi}(G) = \min\{k \mid (G, k) \in \Pi\}.$$

If there is no k such that $(G, k) \in \Pi$, we put $OPT_{\Pi}(G) = +\infty$.
For a vertex-subset maximization problem Π,

$$OPT_{\Pi}(G) = \max\{k \mid (G, k) \in \Pi\}.$$

If no k such that $(G, k) \in \Pi$ exists, then $OPT_{\Pi}(G) = -\infty$.

Bidimensionality

Definition (Bidimensional problem)

A vertex subset problem Π is *bidimensional* if it is contraction-closed, and there exists a constant $c > 0$ such that $OPT_\Pi(\Gamma_k) \geq ck^2$.

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Vertex Cover, Independent Set, Feedback Vertex Set, Induced Matching, Cycle Packing, Scattered Set for fixed value of d, k-Path, k-cycle, Dominating Set, Connected Dominating Set, Cycle Packing, r-Center...

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Lemma (Parameter-Treewidth Bound)

Let Π be a bidimensional problem. Then there exists a constant α_{Π} such that for any connected planar graph G, $\mathbf{tw}(G) \leq \alpha_{\Pi} \cdot \sqrt{OPT_{\Pi}(G)}$. Furthermore, there exists a polynomial time algorithm that for a given G constructs a tree decomposition of G of width at most $\alpha_{\Pi} \cdot \sqrt{OPT_{\Pi}(G)}$.

Bidimensionality: Summing up

Theorem

Let Π be a bidimensional problem such that there exists an algorithm for Π with running time $2^{O(t)}n^{O(1)}$ when a tree decomposition of the input graph G of width t is given. Then Π is solvable in time $2^{O(\sqrt{k})}n^{O(1)}$ on connected planar graphs.

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- Polynomial dependence on n can be turned into linear, so all bidimensionality based algorithms run in time $2^{\mathcal{O}(\sqrt{k})}n$.

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- Planarity is used only to exclude a grid. Thus all the arguments extend to classes of graphs with a similar property.
- Bidimensionality+Separability+$\mathbf{MSO}_2$ brings to Linear kernelization on apex-minor-free graphs. For minor-closed problems to minor-free graphs.