Advanced Treewidth DPs

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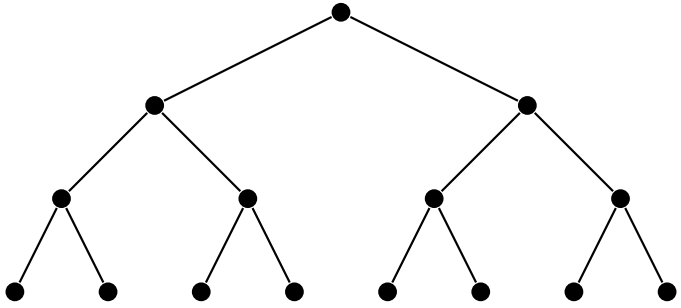
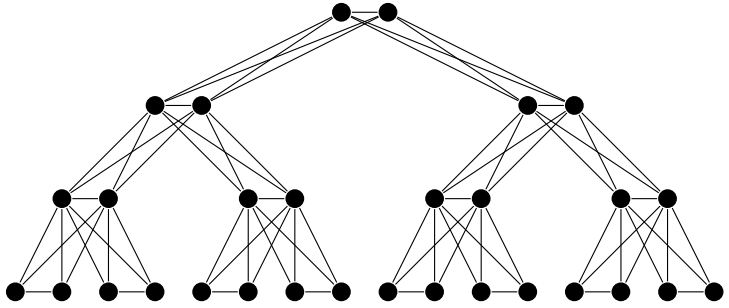
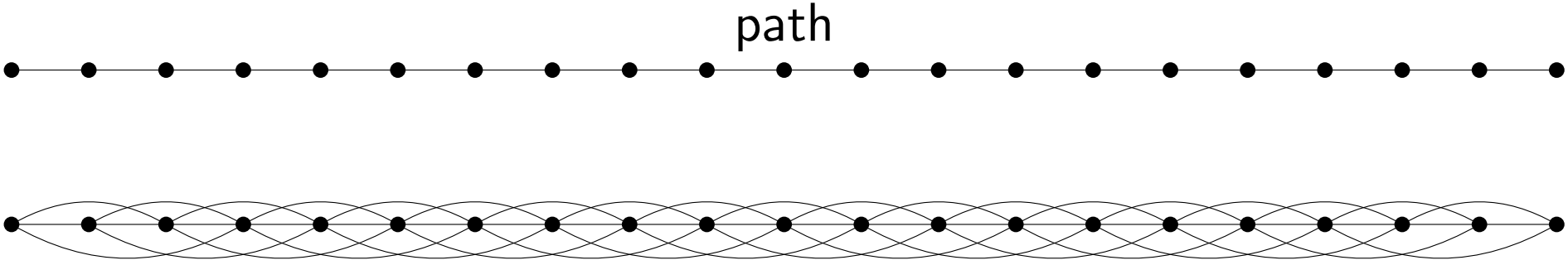


tree



treelike

Treewidth/pathwidth


path
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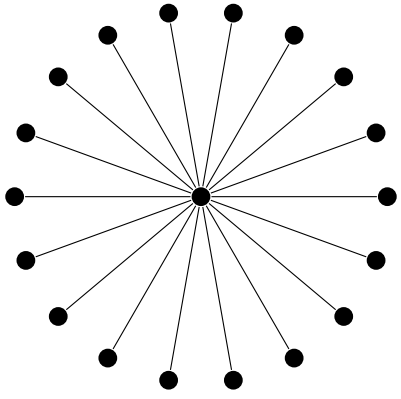
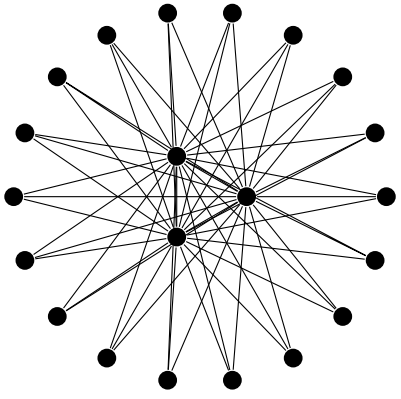


tree

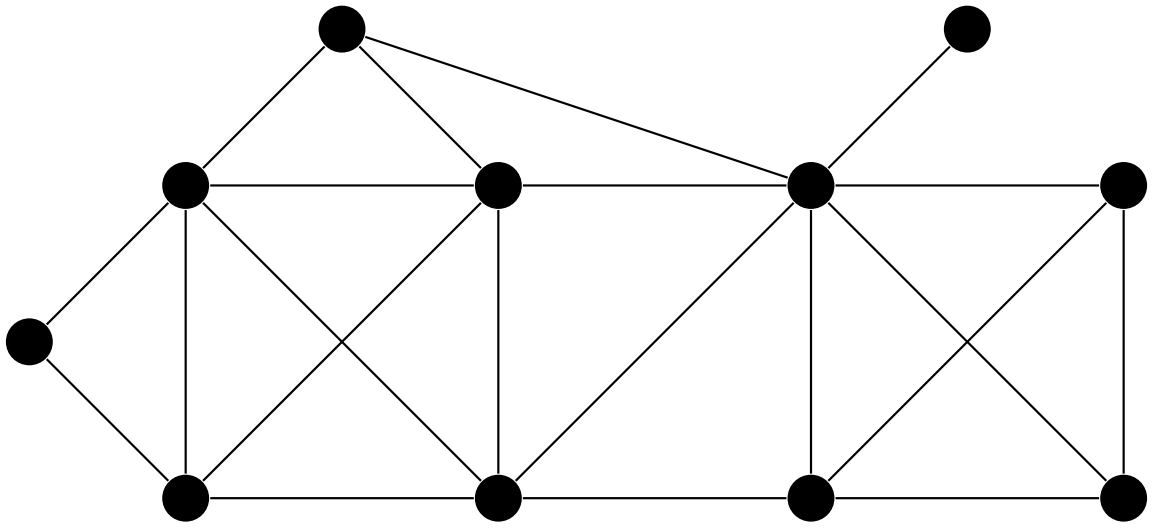


treelike

1. Join nodes for Dominating Set
2. Connectivity problems
    1. Naive algorithm for Hamiltonian Cycle
    2. Cut and count
    3. Rank based approach

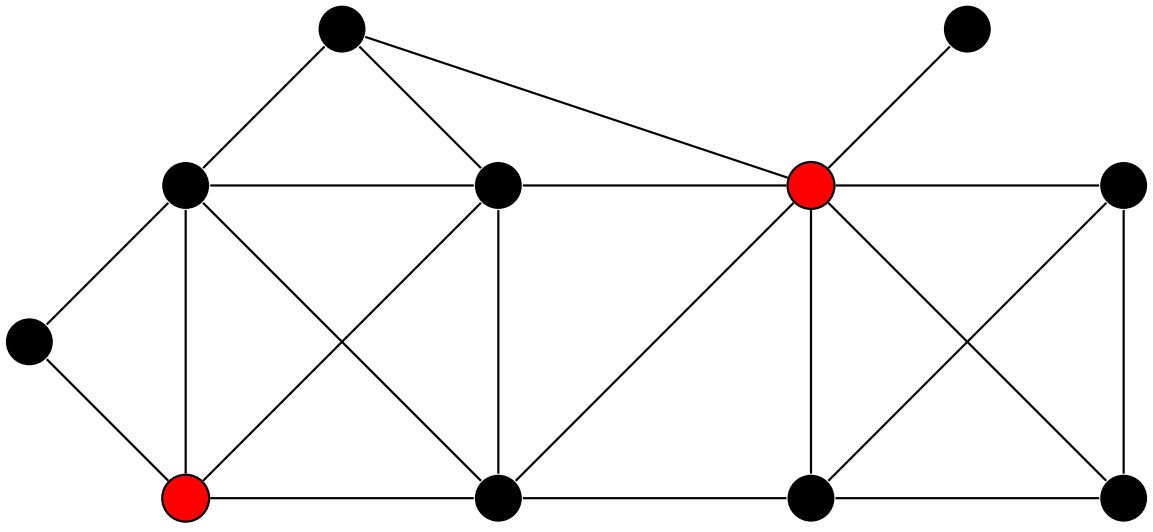
Outline

Dominating Set



Given a graph G, find minimum $X \subseteq V(G)$ such that $N[X] = V(G)$.

Dominating Set



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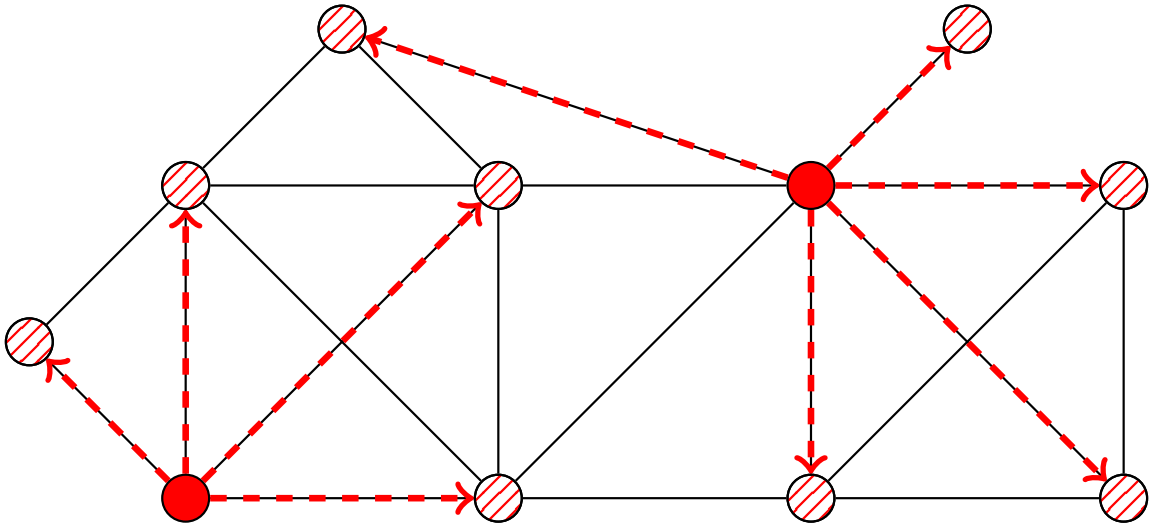

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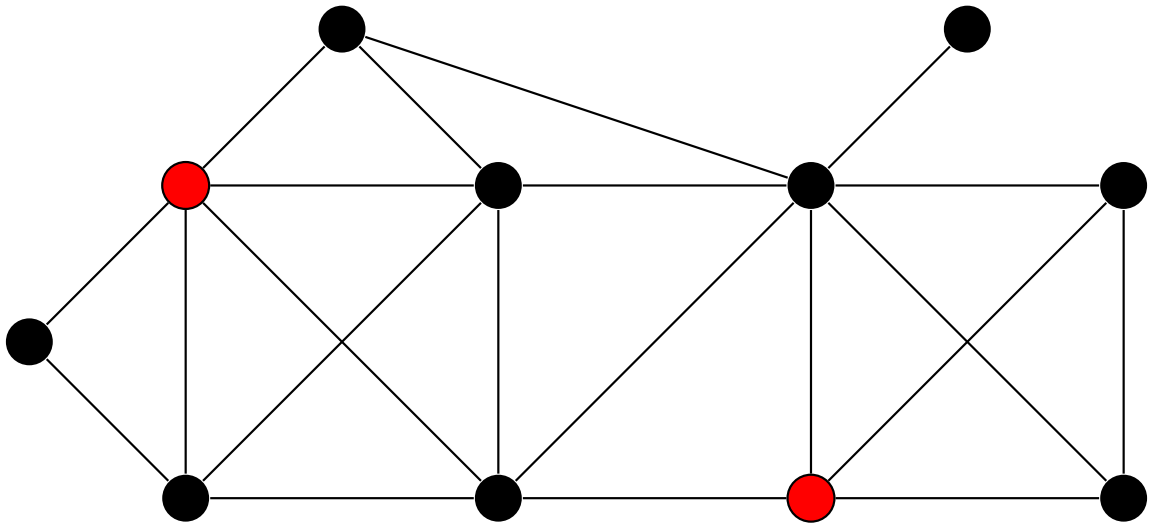

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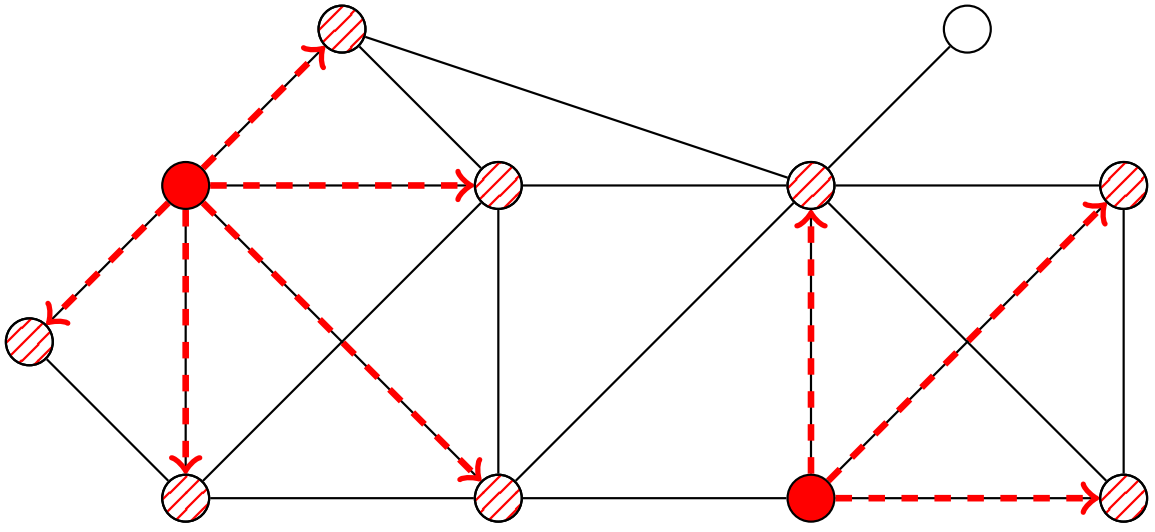

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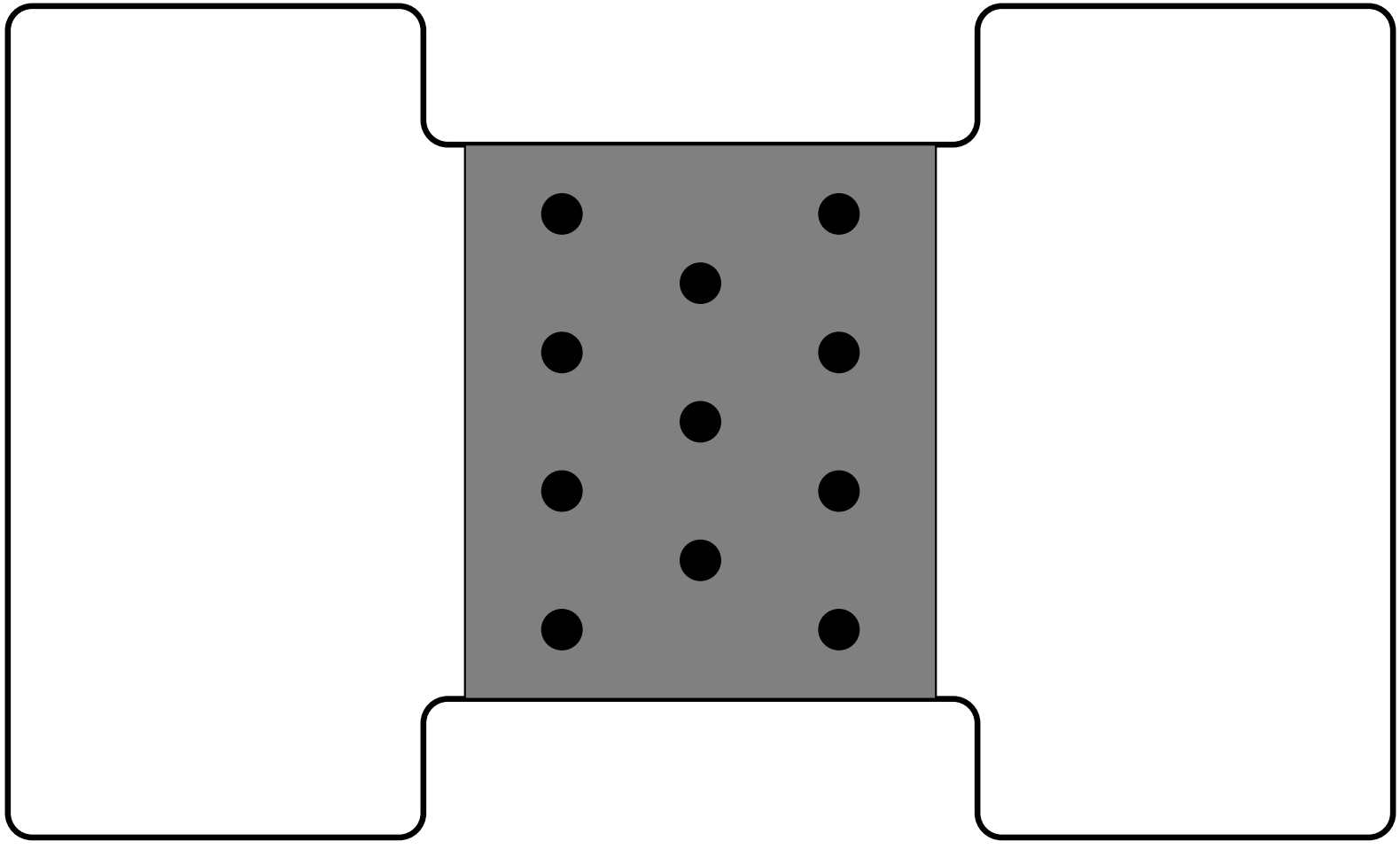
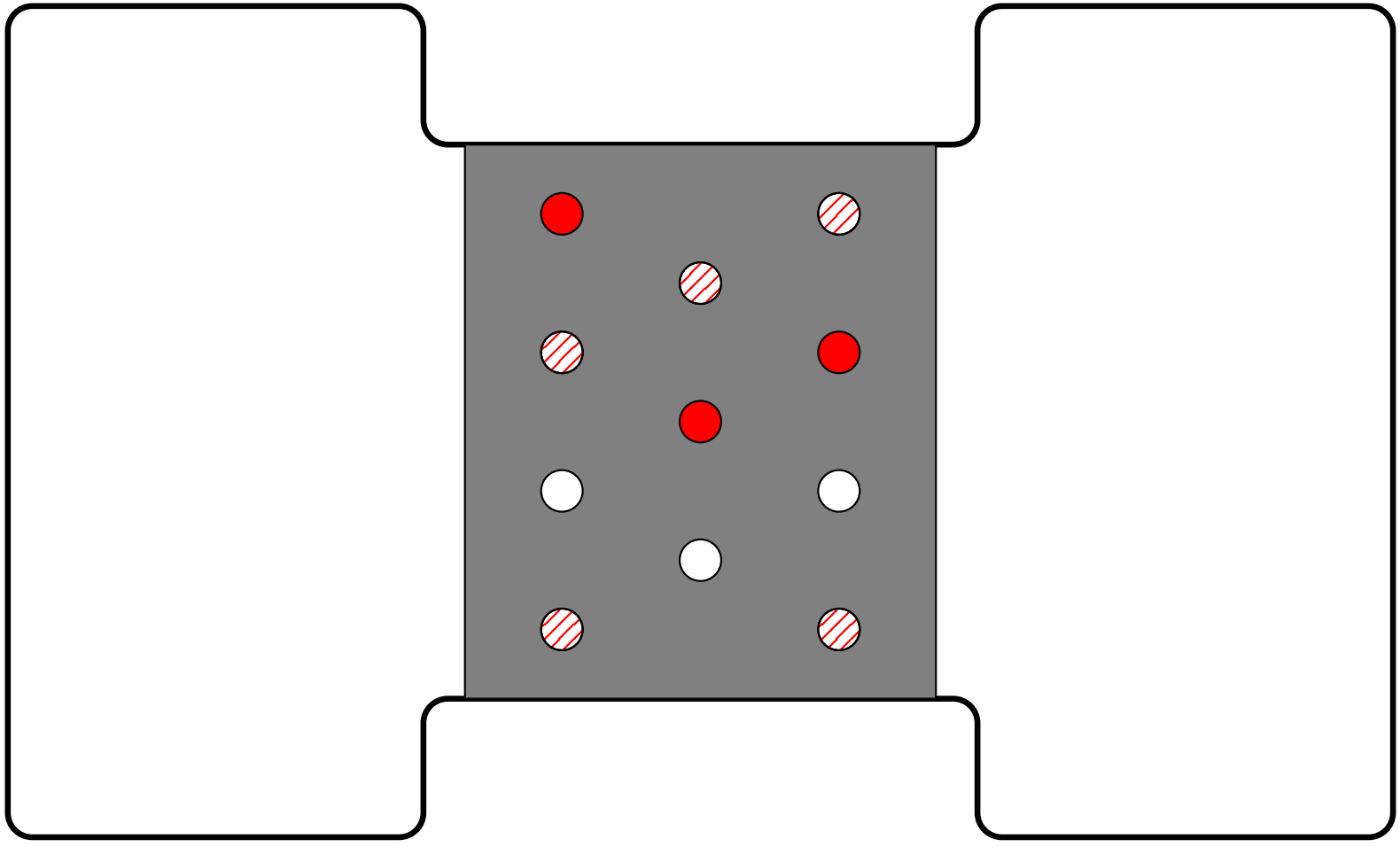


Dominating Set on a separator



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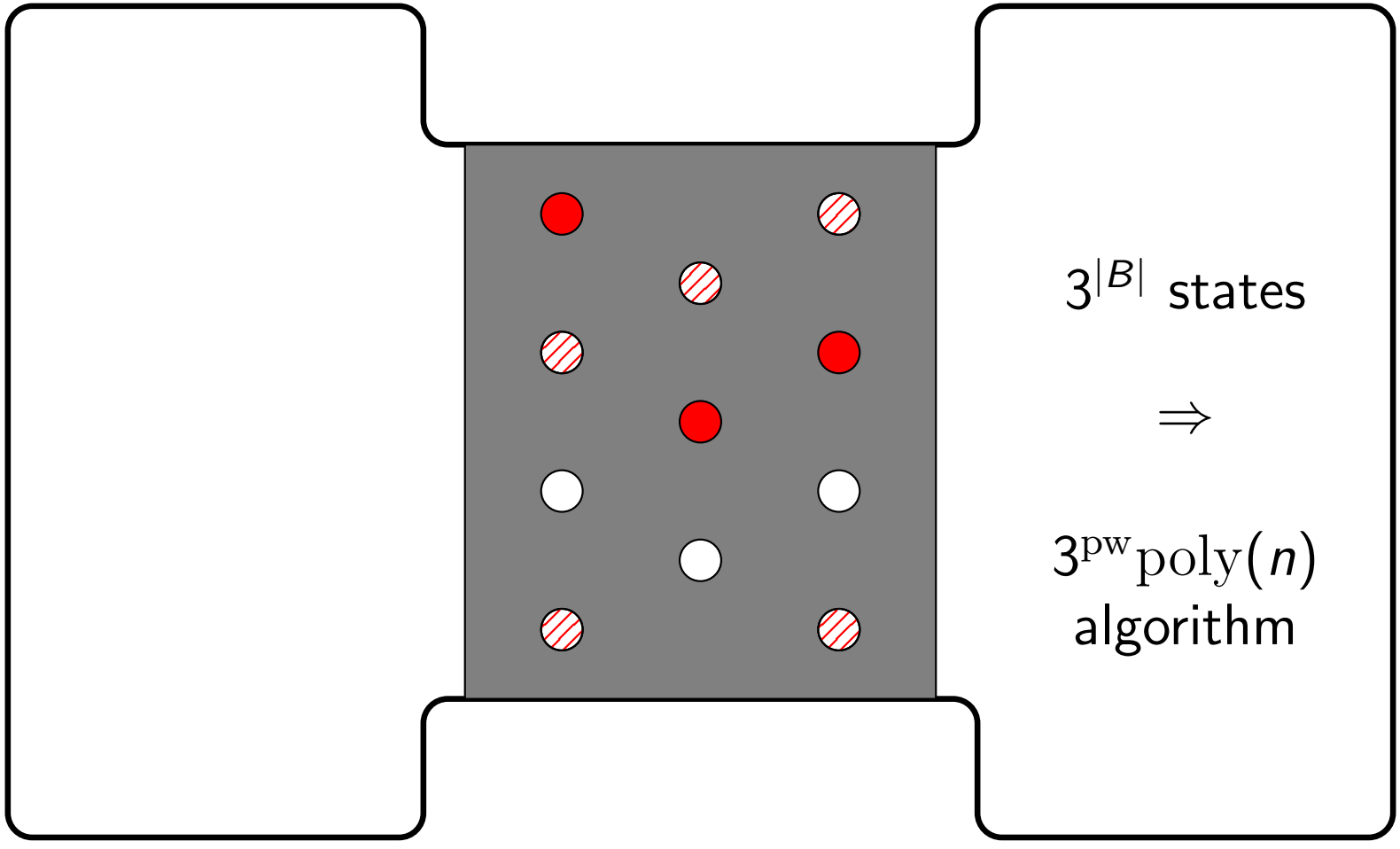

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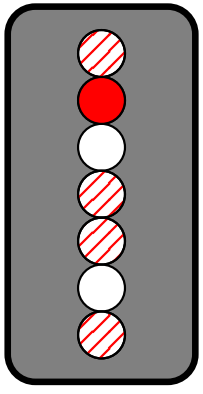
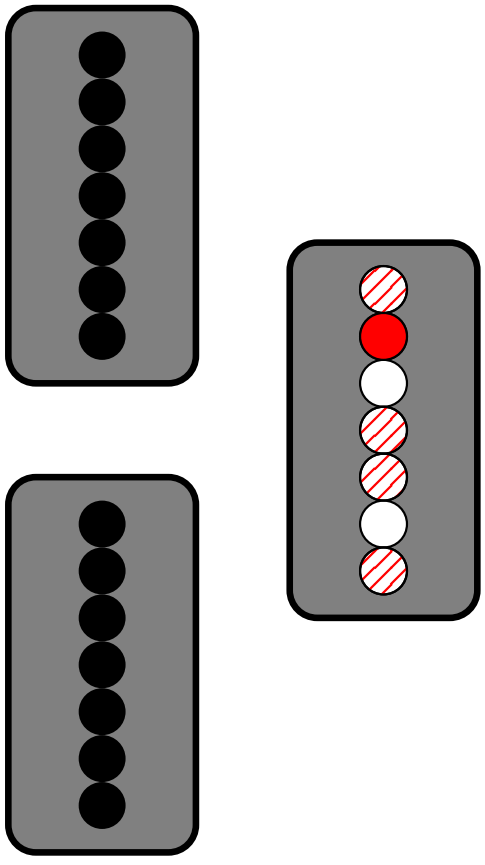


- $B = B_X \uplus B_+ \uplus B_-$;
  $T[B_X, B_+] = \min\{|X| : X \cap B = B_X,$
  $N_G[X] = B_X \cup B_+ \cup \text{forgotten vrts}\}$

Join node



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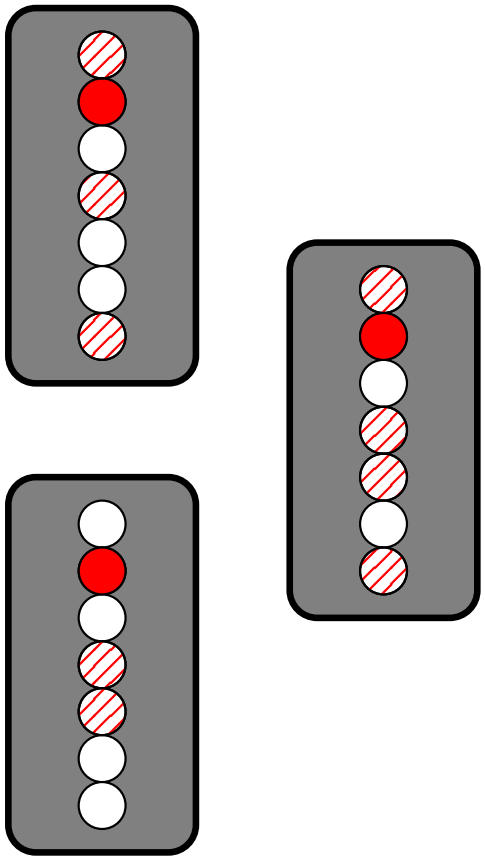

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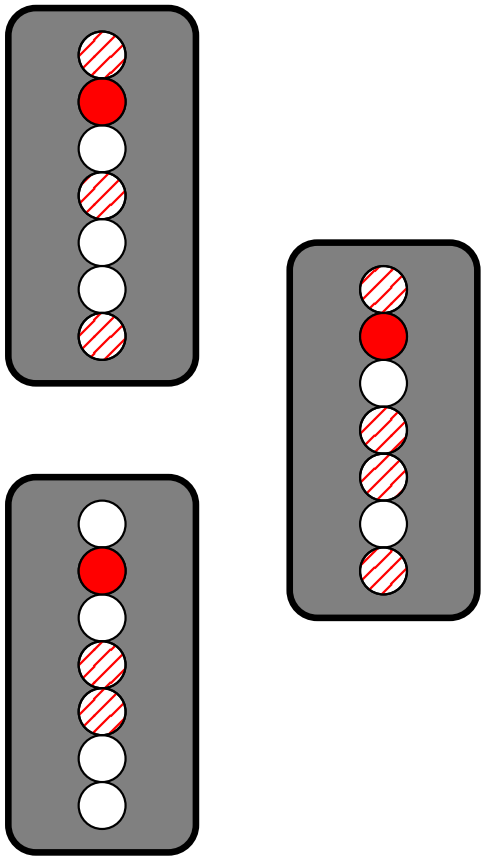

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- $T[B_X, B_+] =$
  $\min\{ T^1[B_X, P] + T^2[B_X, Q] - |B_X| :$
  $P, Q \subseteq B_+, P \cup Q = B_+\}$

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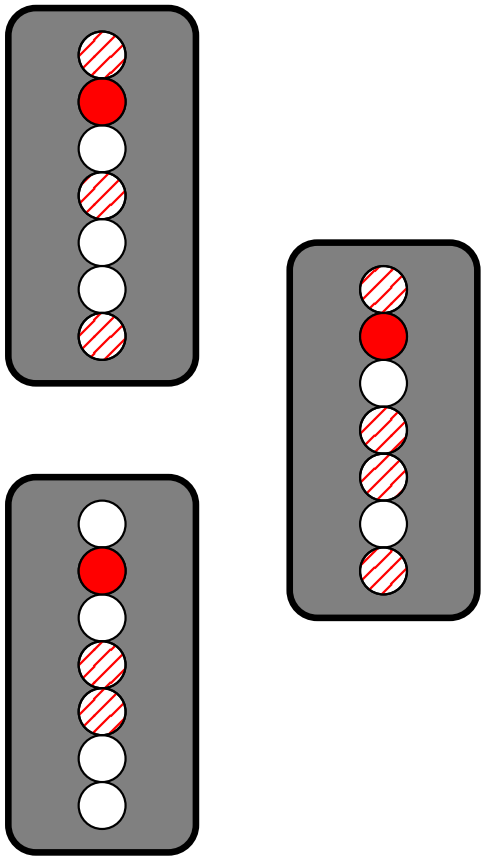

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- naive implementation: $5^{|B|}$ time

Join node

- Need to compute in $2^n \mathrm{poly}(n)$ time convolution

$$f *_{\min} g(A) = \min\{f(B) + g(C) : B \cup C = A\}.$$

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- Define $f_b(B) = [f(B) = b]$, $g_c(C) = [g(C) = c]$.
- Compute covering product and check if $f_b *_c g_c(A) > 0$.

$$f_b *_c g_c(A) = \sum_{B,C \subseteq A : B \cup C = A} f_b(B) g_c(C).$$

For fixed B_X, define $f(P) = T^1[B_X, P]$ and $g(Q) = T^2[B_X, Q]$ over universe $B \setminus B_X$.

Theorem (BRR, ESA'09)

We can solve Dominating Set *in* $3^{\mathrm{tw}}\mathrm{poly}(n)$ *time.*

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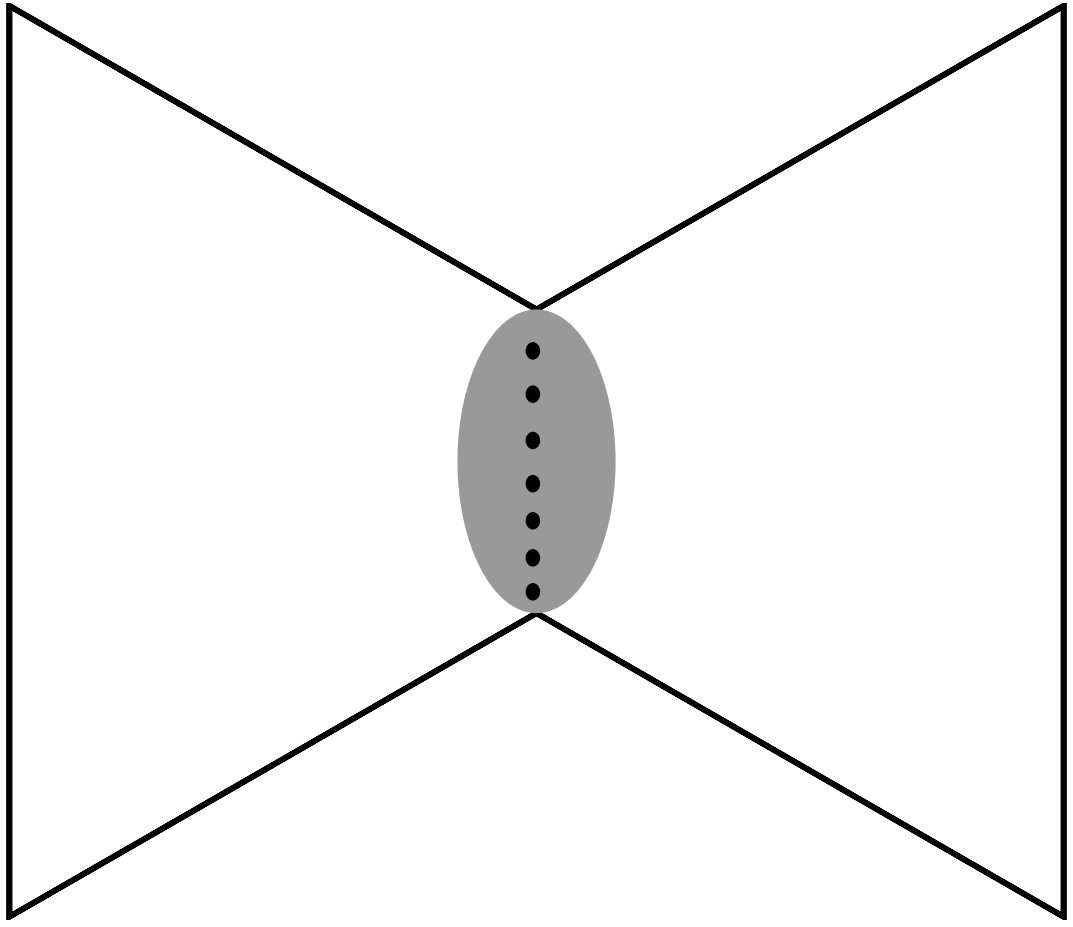
In many cases we can handle the join node efficiently using some convolution.

Connectivity problems

Consider the Hamiltonian cycle problem:



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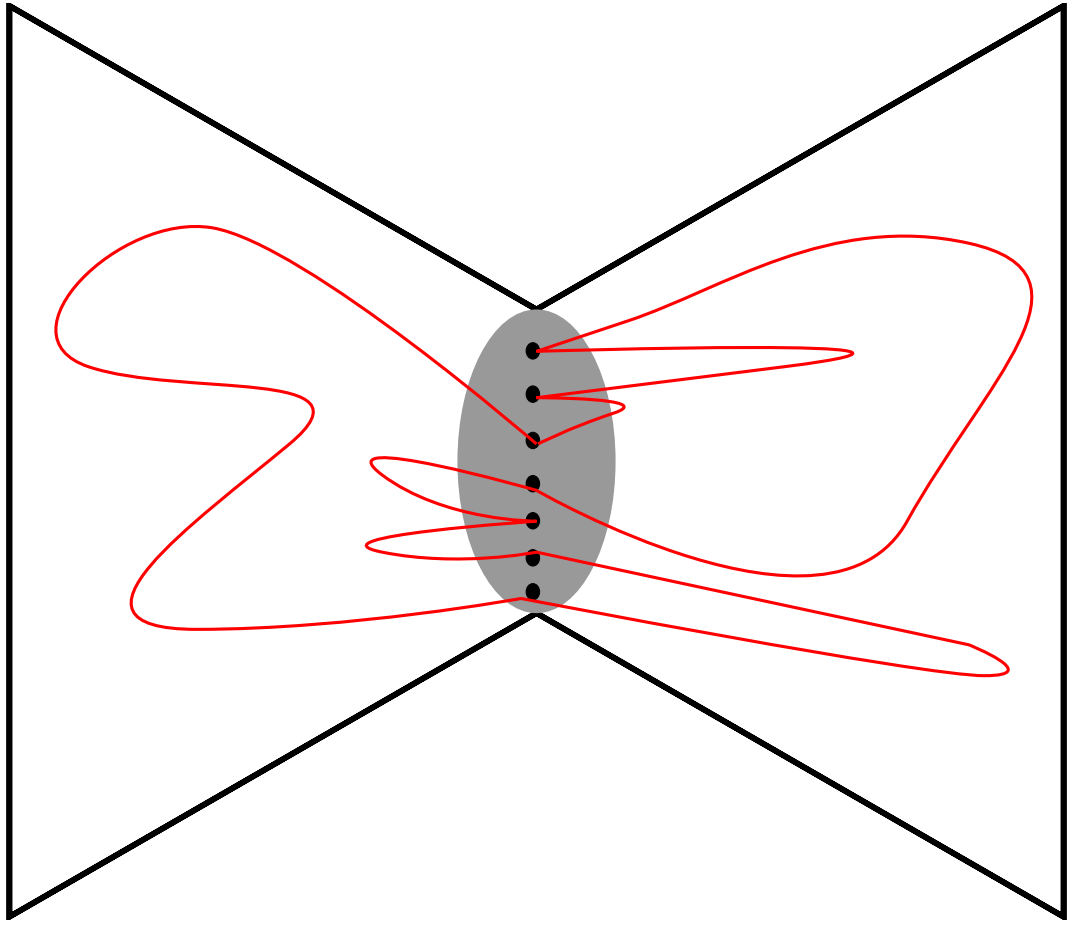

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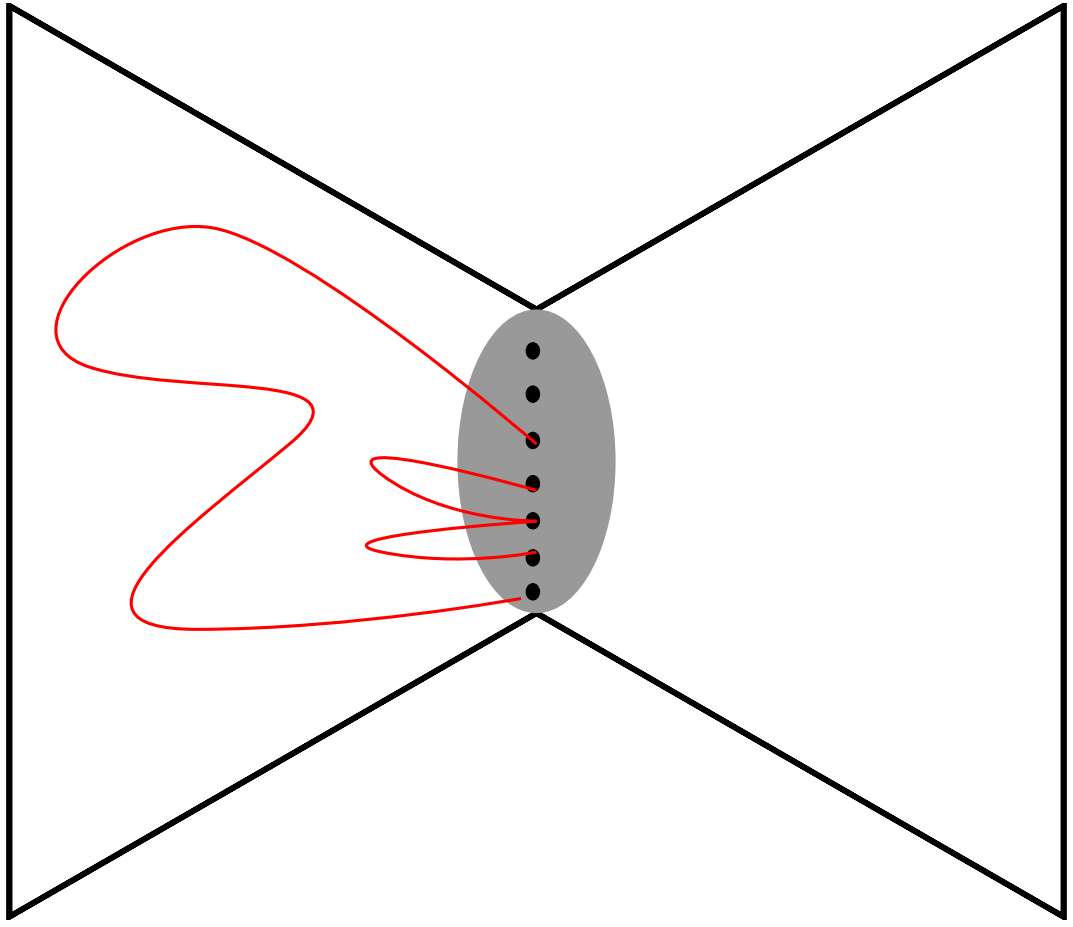
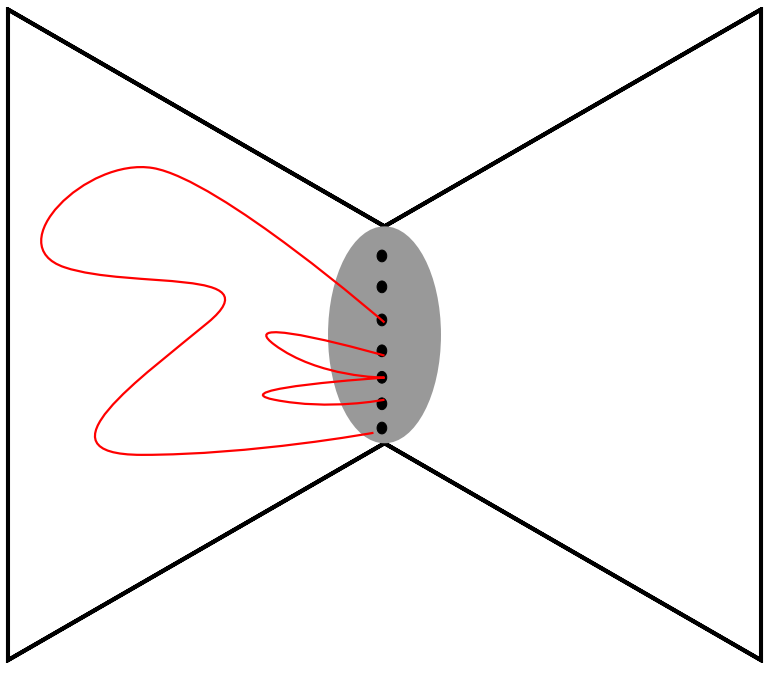


In a state of the standard DP for HC we store:



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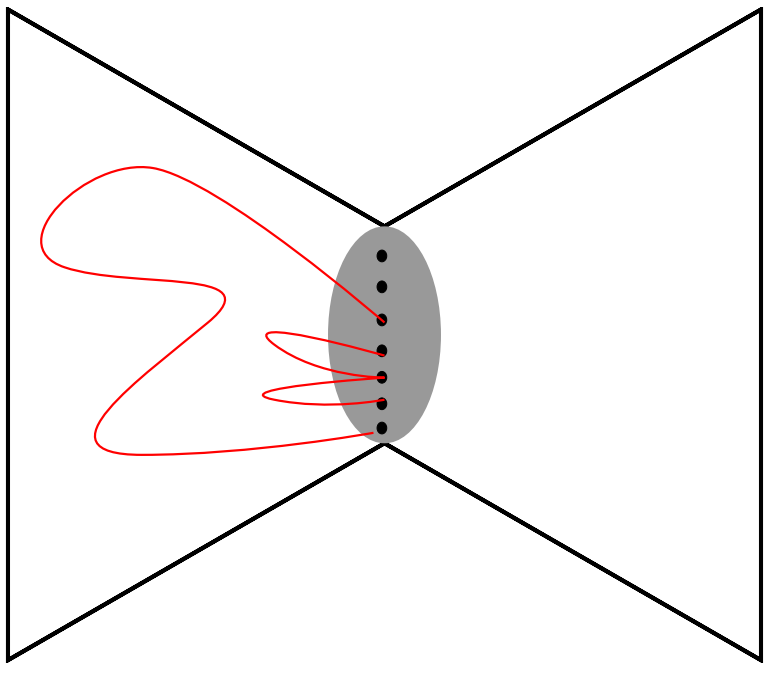

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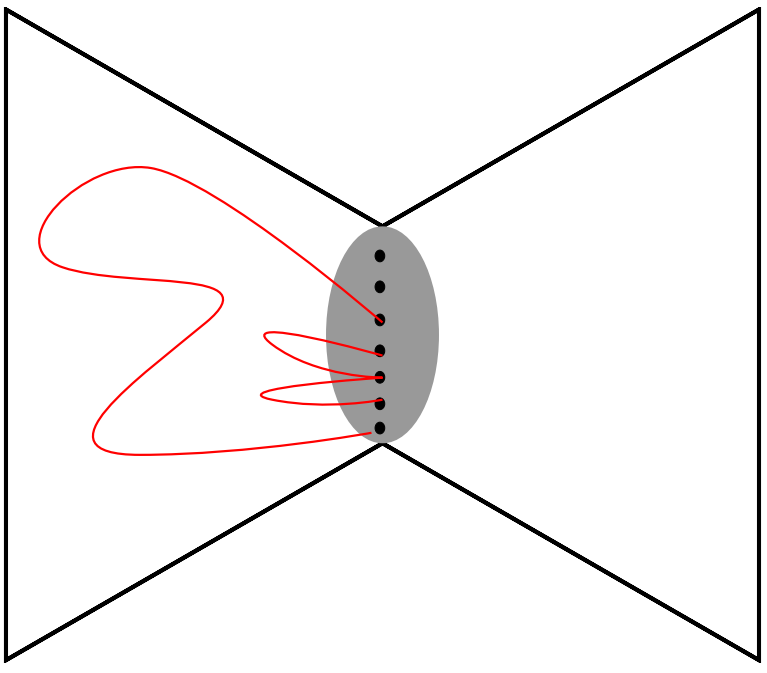
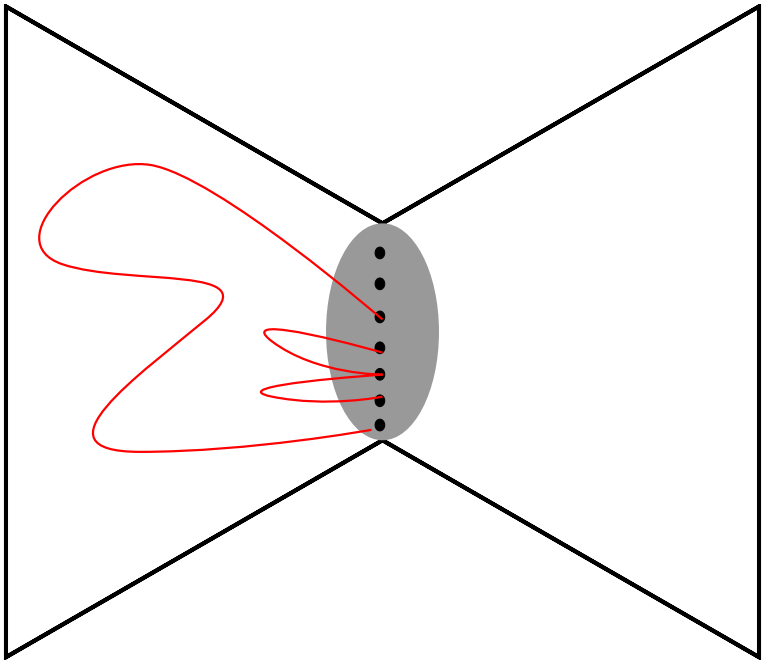


In a state of the standard DP for HC we store:

1. a partition according to degrees $B = B_0 \uplus B_1 \uplus B_2$, (cheap: $3^{|B|}$)
2. a perfect matching on B_1 (path's endpoints).
   (expensive: $|B_1|!! \geqslant (|B_1|/2)!$)



For several problems with "connectivity" flavour standard approach leads to $\mathrm{tw}^{\mathcal{O}(\mathrm{tw})}\mathrm{poly}(n) = c^{\mathrm{tw}\log\mathrm{tw}}\mathrm{poly}(n)$ time algorithms:

- Hamiltonicity,
- Steiner Tree,
- Connected Vertex Cover,
- Connected Dominating Set,
- Feedback Vertex Set,
- Cycle Packing,
- ...

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Better algorithm for special graph classes (planar, H-minor free) using Sphere-Cut Decompositions and Catalan Structures [DPBF, ESA'05 + SODA'08]

Cut&Count technique introduced in [CNPPRW, FOCS 2011] gives Monte Carlo algorithms:

Problem name	algorithm	l.bound (SETH)
Steiner Tree	3^{tw}	$(3-\epsilon)^{\mathrm{tw}}$
Feedback Vertex Set	3^{tw}	$(3-\epsilon)^{\mathrm{tw}}$
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Cycle Packing		$2^{\Omega(p \log p)}$

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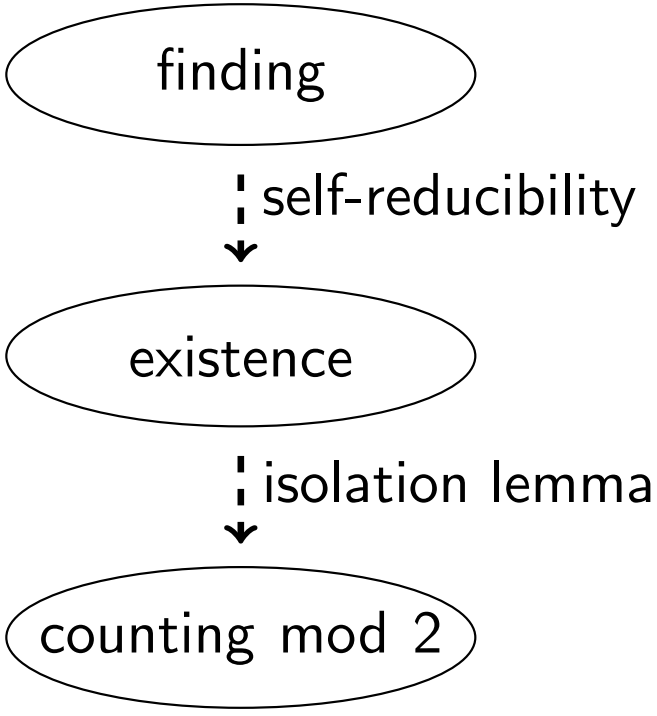
- Lets try once again: a Hamiltonian cycle is a cycle cover that is connected.
- Cycle cover is a subset of edges such that each vertex is of degree exactly two.

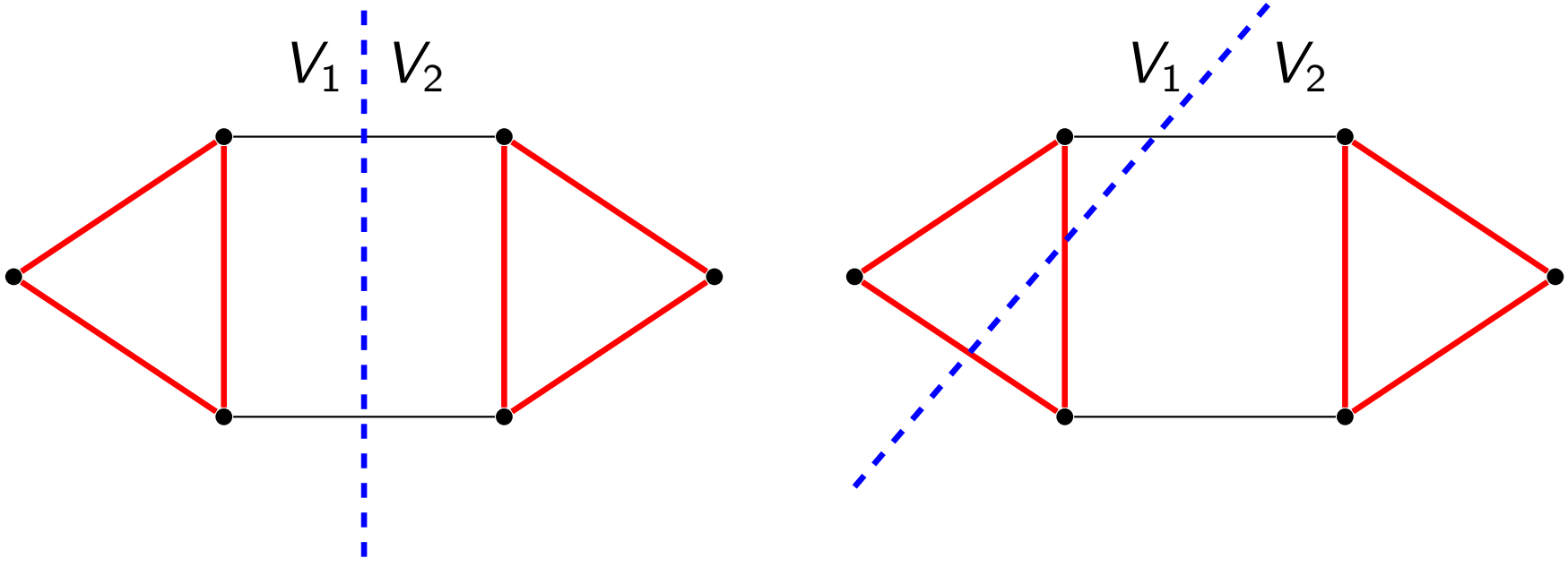

finding
self-reducibility
existence
isolation lemma
counting mod 2

Isolation Lemma[Mulmuley, Vazirani, Vazirani'87]

Let S be the set of Hamiltonian cycles ($S \subseteq 2^E$) with $|\mathcal{S}| > 0$. For each $e \in E$, choose a weight $\omega(e) \in \{1, 2, \ldots, 2|E|\}$ uniformly and independently at random. Then with probability at least $1/2$ there exists exactly one Hamiltonian cycle $X_0 \in \mathcal{S}$ of minimum weight w.r.t. ω.

New goal: find the number of (weighted) Hamiltonian cycles mod 2.

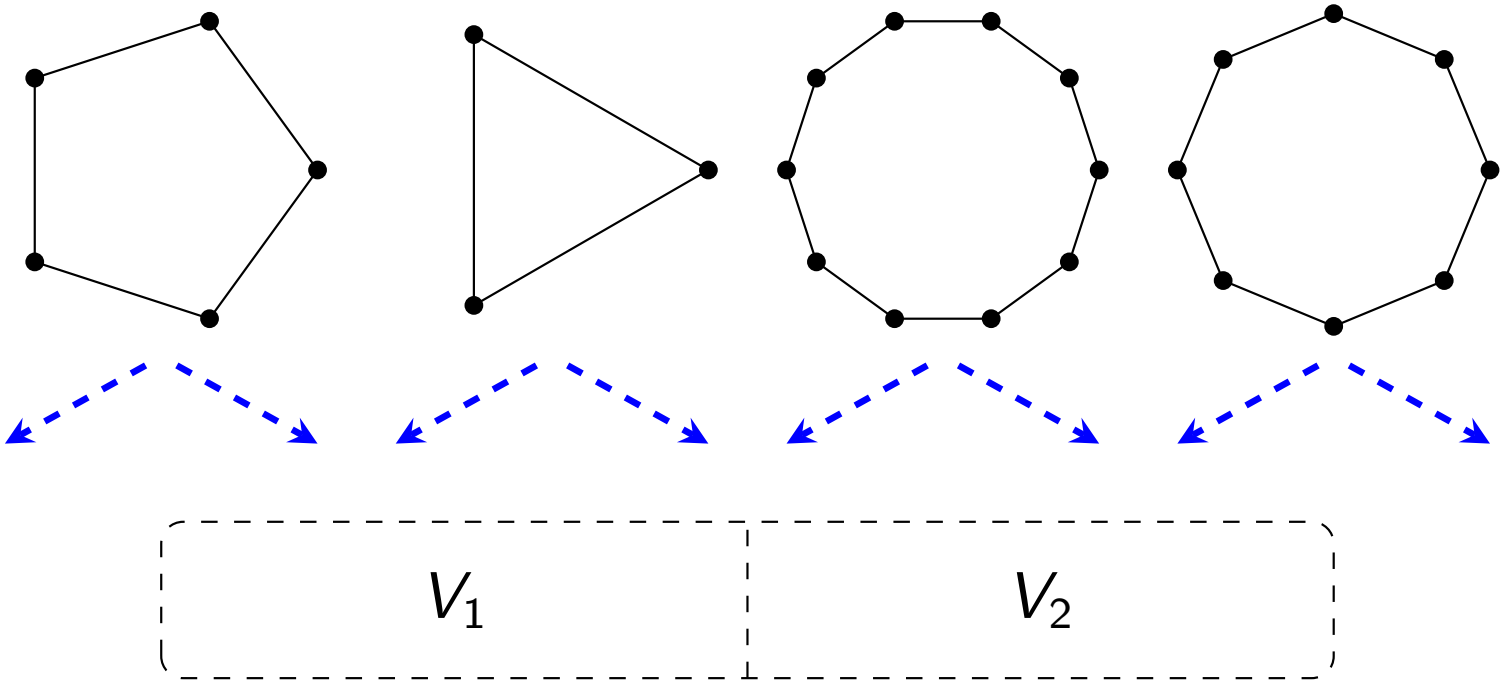
- By a cut (V_1, V_2) we denote a partition $V = V_1 \uplus V_2$.
- We say that a cycle cover $X \subseteq E$ is *consistent* with a cut (V_1, V_2) iff $X \cap E(V_1, V_2) = \emptyset$.



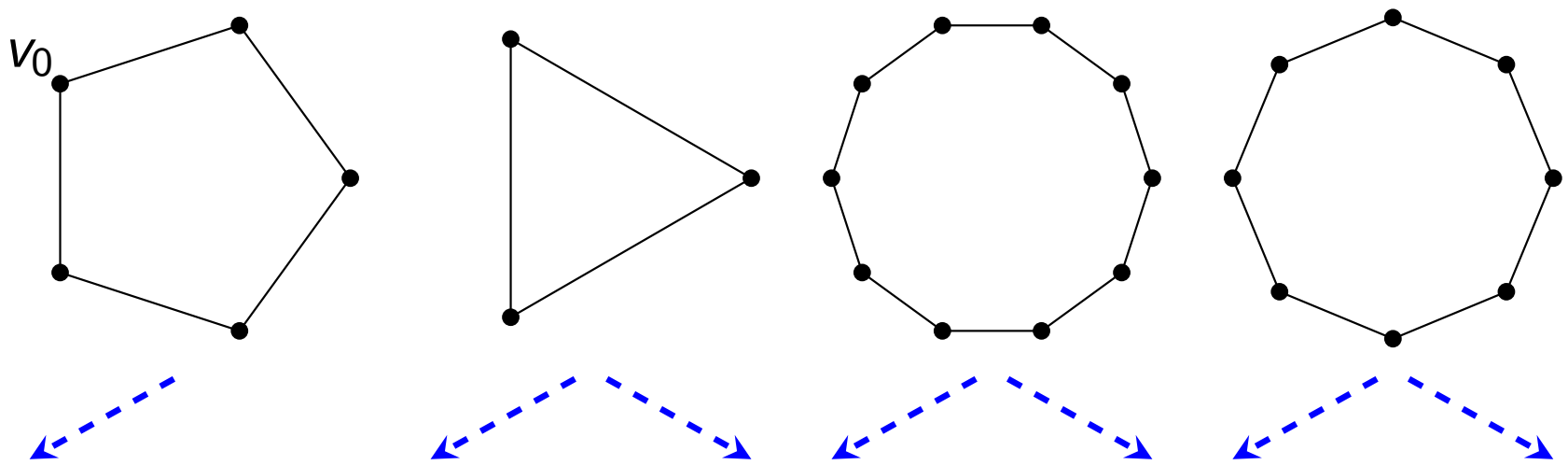
Main idea

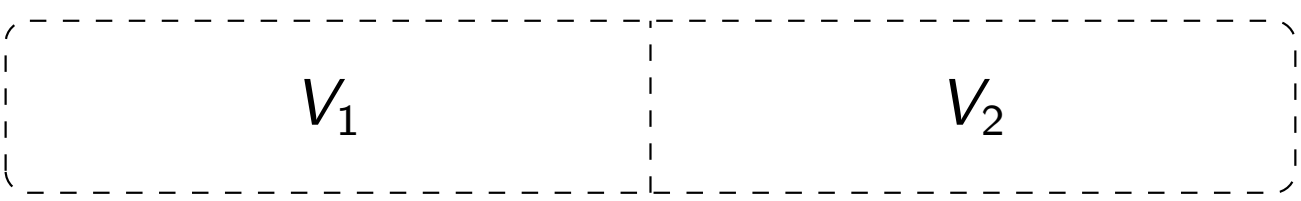
Instead of counting Hamiltonian cycles we will count pairs (cycle cover, consistent cut).

- A cycle cover with a cycles is consistent with exactly 2^a cuts.



- We would like a Hamiltonian cycle to be counted once, while a cycle cover with $\geqslant 2$ cycles an even number of times.
- Fix $v_0 \in V$ and consider cuts with $v_0 \in V_1$.





Cut & count

$$
\begin{aligned}
S = \quad & \text{Hamiltonian cycles} \\
R = \quad & \text{cycle covers} \\
C = \quad & \{(X, (V_1, V_2)) \ : \ X \in R \ \wedge \ v_0 \in V_1 \\
& \qquad \wedge \ X \text{ is consistent with } (V_1, V_2)\}
\end{aligned}
$$

Lemma

$$|C| \equiv |S| \pmod 2$$

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Lemma

$$|C| \equiv |S| \pmod 2$$

One can compute $|C|$ in $6^{\mathrm{tw}}\mathrm{poly}(n)$ time by standard methods!

Cut&Count technique introduced in [CNPPRW, FOCS 2011] gives Monte Carlo algorithms:

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Rank based approach

Although Cut&Count allowed to break the $c^{\mathrm{tw}\log\mathrm{tw}}$ barrier and in some cases even gave (probably) optimal constants c, it has drawbacks:

1. it is randomized,
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[BCKN, ICALP'13][CKN, STOC'13]

There is a way to partially resolve the above issues.

[FLS, SODA'14]

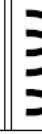
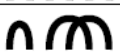
Alternative explanation using matroids.

In a state of the standard DP for HC we store:

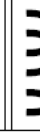
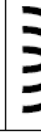
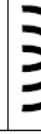
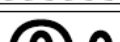
1. a partition according to degrees $B = B_0 \uplus B_1 \uplus B_2$, (cheap)
2. a perfect matching on B_1 (path's endpoints). (expensive)

Main objective

We want to shrink the set of reachable states, by keeping for a fixed partition $B = B_0 \uplus B_1 \uplus B_2$ at most $2^{|B_1|}$ perfect matchings on B_1.

Nr		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
																
1		0	0	0	0	1	1	0	1	1	1	1	0	1	1	0
2		0	0	0	1	0	1	1	1	0	0	1	1	1	0	1
3		0	0	0	1	1	0	1	0	1	1	0	1	0	1	1
4		0	1	1	0	0	0	0	1	1	1	0	1	1	0	1
5		1	0	1	0	0	0	1	0	1	0	1	1	1	1	0
6		1	1	0	0	0	0	1	1	0	1	1	0	0	1	1
7		0	1	1	0	1	1	0	0	0	0	1	1	0	1	1
8		1	1	0	1	0	1	0	0	0	1	0	1	1	1	0
9		1	0	1	1	1	0	0	0	0	1	1	0	1	0	1
10		1	0	1	1	0	1	0	1	1	0	0	0	0	1	1
11		1	1	0	0	1	1	1	0	1	0	0	0	1	0	1
12		0	1	1	1	1	0	1	1	0	0	0	0	1	1	0
13		1	1	0	1	1	0	0	1	1	0	1	1	0	0	0
14		1	0	1	0	1	1	1	1	0	1	0	1	0	0	0
15		0	1	1	1	0	1	1	0	1	1	1	0	0	0	0

From DP to rank - big table

Nr		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
																
1		0	0	0	0	1	1	0	1	1	1	1	0	1	1	0
2		0	0	0	1	0	1	1	1	0	0	1	1	1	0	1
3		0	0	0	1	1	0	1	0	1	1	0	1	0	1	1
4		0	1	1	0	0	0	0	1	1	1	0	1	1	0	1
5		1	0	1	0	0	0	1	0	1	0	1	1	1	1	0
6		1	1	0	0	0	0	1	1	0	1	1	0	0	1	1
7		0	1	1	0	1	1	0	0	0	0	1	1	0	1	1
8		1	1	0	1	0	1	0	0	0	1	0	1	1	1	0
9		1	0	1	1	1	0	0	0	0	1	1	0	1	0	1
10		1	0	1	1	0	1	0	1	1	0	0	0	0	1	1
11		1	1	0	0	1	1	1	0	1	0	0	0	1	0	1
12		0	1	1	1	1	0	1	1	0	0	0	0	1	1	0
13		1	1	0	1	1	0	0	1	1	0	1	1	0	0	0
14		1	0	1	0	1	1	1	1	0	1	0	1	0	0	0
15		0	1	1	1	0	1	1	0	1	1	1	0	0	0	0

From DP to rank - big table

Theorem

The rank of this matrix over $\mathbb{Z}_2$ is exactly $2^{|B|/2-1}$.

- Consider a matrix $\mathbf{T}$:
  - rows indexed by matchings M,
  - colums indexed by cuts (V_1, V_2) with $v_0 \in V_1$,
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- Consider a matrix $\mathbf{T}$:
  - rows indexed by matchings M,
  - colums indexed by cuts (V_1, V_2) with $v_0 \in V_1$,
  - 1 if M is consistent with the cut, 0 otherwise.
- Crucial observation: $\mathbf{M} = \mathbf{T}\mathbf{T}^\top$ over $\mathbb{Z}_2$.
  - In a cell (M_1, M_2), the product on the left counts the number of cuts consistent with both M_1 and M_2.
  - Two matchings sum up to Hamiltonian cycle if there is only one such cut, and an even number of such cuts if they do not.

- We can keep only realizable matchings that are independent in $\mathbf{M} \Rightarrow$ only $2^{|B|/2-1}$ of them.

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- Using $\mathbf{T}$, we can efficiently discard redundant (dependent) matchings.
- Discard heaviest dependent matching $\Rightarrow$ can solve weighted variants.
- Generalizes to all Cut & Count algorithms.

Weighted Steiner Tree	$n(1+2^{\omega+1})^{\mathrm{tw}}\mathrm{tw}^{\mathcal{O}(1)}$
Traveling Salesman	$n(5+2^{(\omega+2)/2})^{\mathrm{tw}}\mathrm{tw}^{\mathcal{O}(1)}$
k-Path	$n(5+2^{(\omega+2)/2})^{\mathrm{tw}}(k+\mathrm{tw})^{\mathcal{O}(1)}$
Feedback Vertex Set	$n(1+2^{\omega+1})^{\mathrm{tw}}\mathrm{tw}^{\mathcal{O}(1)}$

Conclusions and open problems

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5. Exercises: all from Chapter 11.